

# Scale invariant theory of gravity and the standard model of particles

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In this paper we demonstrate how local scale invariance – invariance under Weyl rescalings – may safely coexist with broken electroweak symmetry in our present Universe. It is required that Weyl’s geometric theory governs the affine structure of spacetime. We discuss the consequences of the resulting scale invariant theory of gravity and particles for high-energy physics and cosmology. We found that nothing besides scale invariance and cosmological inflation is required to explain the large hierarchy between the Higgs and the Planck masses. In the present setup the late-time speedup of the cosmic expansion can be explained without the need for the dark energy. Moreover, the observational evidence on accelerated expansion can be explained even if the Universe is not expanding at all. The gauge degree of freedom which is distinctive of the present scale invariant setting leads to a picture which shares certain resemblance with the multiverse scenario.

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## I. INTRODUCTION

The apparently straightforward statements made by Dicke in Ref. [1] about (i) the naturalness of requiring invariance of the laws of physics under transformations of units, and (ii) the fact that there may be more than one feasible way of establishing the equality of units at different spacetime points, raise the question about considering generalizations of (pseudo)Riemann geometry. The first (and simplest) such generalization that comes to one’s mind is Weyl geometry [2–12] (for an historical perspective see [13]). Weyl’s geometric theory is no more than a generalization of Riemann geometry to include point dependent length of vectors during parallel transport, in addition to the point dependent property of vectors directions. It is assumed that the length of a given vector  $\mathbf{l}$  ( $l \equiv \sqrt{g_{\mu\nu} l^\mu l^\nu}$ ) varies from point to point in spacetime according to:  $dl = l w_\mu dx^\mu / 2$ , where  $w_\mu$  is the Weyl gauge boson. Hence, the second of the Dicke’s statements above on point-dependent units of length, finds a natural realization within Weyl geometry. The first of the statements made by Dicke – see also [14, 15] – can be implemented in any theory of gravity which is invariant under the Weyl rescalings/local scale transformations:<sup>1</sup>

$$g_{\mu\nu} \rightarrow \Omega^{-2} g_{\mu\nu}, \quad w_\mu \rightarrow w_\mu + 2\partial_\mu \ln \Omega, \quad (1)$$

where the (smooth) positive spacetime function  $\Omega^2 = \Omega^2(x)$  is the conformal factor, and the spacetime coincidences/coordinates ( $x \equiv \{x^0, \mathbf{x}\}$ ) are kept unchanged.

A question then arises: can be general relativity (GR) coupled to the standard model of particles (SMP) com-

patible with scale invariance? The answer is affirmative.<sup>2</sup> In the references [9, 11, 12, 18] this goal was achieved by means of different approaches.

The setup of [9] is given by the following action:

$$S = \int d^4x \sqrt{|g|} \left[ \frac{\xi \phi^2}{2} R^{(w)} + \frac{1}{2} (D\phi)^2 - \frac{1}{4g^2} H_{\mu\nu} H^{\mu\nu} - \frac{\lambda}{24} \phi^4 + \dots \right], \quad (2)$$

where  $(D\phi)^2 \equiv g^{\mu\nu} D_\mu \phi D_\nu \phi$ ,  $D_\mu = \partial_\mu - w_\mu/2$  is the gauge covariant derivative, the “...” account for the terms quadratic in the curvature, and the field strength of the Weyl gauge boson  $w_\mu$  is defined as

$$H_{\mu\nu} := \partial_\mu w_\nu - \partial_\nu w_\mu.$$

In the above action the index ( $w$ ) refers to Weylian quantities/operators which are defined in terms of the affine connection of some Weylian manifold:

$$\Gamma_{\alpha\beta}^\mu = \{\mu_{\alpha\beta}\} + \frac{1}{2} \left( \delta_\alpha^\mu w_\beta + \delta_\beta^\mu w_\alpha - g_{\alpha\beta} w^\mu \right), \quad (3)$$

where  $\{\mu_{\alpha\beta}\}$  are the standard Christoffel symbols of the metric (properly the affine connection of the Riemann space). The action (2) is invariant under the Weyl rescalings (1) plus the gauge transformation  $\phi \rightarrow \Omega \phi$ . Here the spontaneous breakdown of local scale invariance is implemented through the additional scalar field  $\phi$  which

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<sup>1</sup> The conformal transformation of the metric in (1) is what Dicke regards as a transformation of units in [1].

<sup>2</sup> In [16] it has been shown that scale invariance is very much related with the effect of asymptotic conformal invariance, where quantum field theory predicts that theory becomes effectively conformal invariant. Meanwhile in Ref. [17] the authors present the most general scalar tensor theories, in four dimensions, consistent with second order field equations which exhibit (local) scale invariance.

acquires a vacuum expectation value (VEV), leaving us with GR coupled to a massive vector field [9].

In a similar fashion in [11] the author reconsiders the so called Weyl-Omote-Dirac action:

$$S = \int d^4x \sqrt{|g|} \left[ \xi \phi^2 R^{(w)} + \frac{1}{2} |D\phi|^2 - \frac{\lambda}{4} |\phi|^4 - \frac{1}{4} H_{\mu\nu} H^{\mu\nu} \right], \quad (4)$$

where  $|D\phi|^2 \equiv g^{\mu\nu} (D_\mu \phi)^\dagger (D_\nu \phi)$ , and the gauge covariant derivative is defined as in (2). As in the former setup the field equations of the theory (4) are invariant under the Weyl rescalings (1) plus  $\phi \rightarrow \Omega \phi$ . In this theory the acquirement of mass arises as a result of coupling to gravity in agreement with the understanding of mass as the gravitational charge of fields.

In [12] the non-minimally coupled scalar field  $\phi$  is identified with the Higgs gauge boson in the unitary gauge  $H^T = (0, h)/\sqrt{2}$ :

$$S = \int d^4x \sqrt{|g|} \left[ \frac{\xi |h|^2}{2} R^{(w)} - \frac{1}{2} |Dh|^2 - \frac{\lambda}{4} (|h|^2 - v_0^2)^2 - \frac{1}{4} (H_{\mu\nu} H^{\mu\nu} + W_{\mu\nu}^k W_k^{\mu\nu} + B_{\mu\nu} B^{\mu\nu}) \right], \quad (5)$$

where

$$|h|^2 \equiv h^\dagger h = h^2, \quad |Dh|^2 \equiv g^{\mu\nu} (D_\mu h)^\dagger (D_\nu h),$$

$W_{\mu\nu}^k$  and  $B_{\mu\nu}$  are the field strengths of the  $SU(2)$  and  $U(1)$  bosons respectively (see the appendix), and  $\xi$  is the non-minimal coupling parameter. In the theory (5) the electroweak (EW) symmetry breaking potential not only allows for generation of masses of the gauge bosons (and fermions) but, also, generates the Planck mass  $M_{\text{Pl}} = \sqrt{\xi} v_0$ , where  $v_0 \approx 246$  GeV, and  $\xi \sim 10^{32} - 10^{34}$  is too large to meet the observational constraints.<sup>3</sup> Before breakdown of scale symmetry, the action (5) is invariant under (1) plus the Higgs field rescaling  $h \rightarrow \Omega h$ . The gauge covariant derivative of the Higgs field in (5) is defined as

$$D_\mu h := (D_\mu^* - w_\mu/2)h, \quad (6)$$

where

$$D_\mu^* h \equiv \left( \partial_\mu + \frac{i}{2} g W_\mu^k \sigma^k + \frac{i}{2} g' B_\mu \right) h, \quad (7)$$

is the gauge covariant derivative in the standard EW theory, with  $W_\mu^k = (W_\mu^\pm, W_\mu^0)$  - the  $SU(2)$  bosons,  $B_\mu$  - the

$U(1)$  boson,  $\sigma^k$  - the Pauli matrices, and  $(g, g')$  - the gauge couplings.

Recall that in this last case, as well as in (2) and in (4), the affine structure of the spacetime is assumed Weylian. This means, in turn, that the units of measure are point-dependent. Consequently, in equations (2), (4) and (5),

$$R_{\alpha\beta\mu\nu}^{(w)}, R_{\mu\nu}^{(w)}, R^{(w)}, \nabla_\mu^{(w)},$$

etc., are the Riemann-Christoffel curvature tensor, the Ricci tensor, the curvature scalar and the covariant derivative operator of the Weyl geometry, respectively. These are defined in terms of the affine connection of the Weyl space (3).

Unlike the above cases, in the approach of [18] (see also [20–22]), which is given by the following action:<sup>4</sup>

$$S = \int d^4x \sqrt{|g|} \left[ \frac{1}{12} (\phi^2 - 2|H|^2) R + \frac{1}{2} (\partial\phi)^2 - |D^* H|^2 + \frac{\lambda'}{4} |\phi|^4 - \frac{\lambda}{4} (|H|^2 - \alpha^2 \phi^2)^2 \right], \quad (8)$$

where  $|H|^2 \equiv H^\dagger H$ ,  $|D^* H|^2 \equiv g^{\mu\nu} (D_\mu^* H)^\dagger (D_\nu^* H)$ , although no specific statement on the geometric structure of the theory is made, pseudo-Riemannian spacetimes are implicitly assumed, just as in GR (for a detailed discussion see the section XI). The action (8) is invariant under the following scale transformations:

$$g_{\mu\nu} \rightarrow \Omega^{-2} g_{\mu\nu}, \quad (\phi, H) \rightarrow \Omega (\phi, H).$$

Here the only Yukawa couplings of the dilaton  $\phi$  allowed by  $SU(3) \times SU(2) \times U(1)$  are to the right-handed neutrinos [18]. In spite of the fact that the theory (8), in similitude with general relativity, operates in pseudo-Riemannian manifolds – unlike (2), (4), and (5), which operate in Weylian manifolds – the setup of [18] (see also [20–22]) shares certain similarity with the ones in Ref. [9, 11, 12]:

- In all of them a certain additional scalar field (identified in [12] with the Higgs boson and in [18] with the dilaton) is non-minimally coupled to gravity, and
- there are no any dimensionful constants, in particular no Einstein-Hilbert (HE) term with its dimensionful Newton's constant.

The latter, as well as other dimensionful parameters, emerge from a single source: the scalar field which is non-minimally coupled to gravity. The only scale is generated by gauge fixing the scalar field to a constant.

<sup>3</sup> The first bound on the value of the non-minimal coupling ( $\xi < 2.6 \times 10^{15}$ ) was derived in [19].

<sup>4</sup> The gauge covariant derivative  $D_\mu^*$  in (8) coincides with the definition given in (7).

From the point of view of [9, 11, 12, 18], since the breakdown of local scale invariance – be it either by means of a Higgs-like mechanism or by gauge fixing the non-minimally coupled scalar field to a constant – is a necessary requirement for the generation of the fundamental scales such as the Newton’s constant, invariance under the local scale transformations (1), or units transformations in Dicke’s sense, in our present Universe seems to be forbidden. Yet one may persist and wonder whether there is any chance whatsoever for local scale invariance to coexist together with breakdown of EW symmetry when the SMP is coupled to gravity. In other words, keeping on the spirit of Dicke’s thoughts when he wrote [1]: “The laws of physics must be invariant under a transformation of units.” one wonders whether scale invariance can be an actual symmetry of our present Universe.

In this paper we will investigate this issue. We shall show that if the Weyl’s geometric theory – in particular a special case of it called as Weyl integrable geometry (WIG) – governs the affine properties of the spacetime, scale invariance and symmetry breaking can be compatible concepts, i. e. local scale invariance and broken EW symmetry may safely coexist together. Our setup differs from those in the references [9, 11, 12, 18, 20–22] in that the fundamental mass scale  $M_{\text{pl}}^2$  is included from the start in the corresponding WIG-EH action<sup>5</sup>

$$S_{\text{EH}}^{(w)} = \frac{1}{2} \int d^4x \sqrt{|g|} M_{\text{pl}}^2 e^\varphi R^{(w)}. \quad (9)$$

Hence, in order for the latter to respect local scale symmetry the Planck mass must enter in the following combination  $M_{\text{pl}}^2(\varphi) = M_{\text{pl}}^2 e^\varphi$  which, under  $g_{\mu\nu} \rightarrow \Omega^{-2} g_{\mu\nu}$ ,  $\varphi \rightarrow \varphi + 2 \ln \Omega$ , transforms like:  $M_{\text{pl}}^2(\varphi) \rightarrow \Omega^2 M_{\text{pl}}^2(\varphi)$ . This is a feasible possibility thanks to the adoption of Weyl integrable geometry – where the measuring units are point dependent – as the theory which correctly describes the affine properties of spacetime. WIG is obtained from Weyl geometry by replacing  $w_\mu \rightarrow \partial_\mu \varphi \Rightarrow H_{\mu\nu} = 0$ . In consequence the Weyl gauge scalar  $\varphi$ , which is non-minimally coupled to the curvature scalar in (9), takes part in the definition of the affine connection of space (compare with Eq. (3)):

$$\Gamma_{\beta\gamma}^\alpha = \{\alpha_\beta\gamma\} + \frac{1}{2} (\delta_\beta^\alpha \partial_\gamma \varphi + \delta_\gamma^\alpha \partial_\beta \varphi - g_{\beta\gamma} \partial^\alpha \varphi). \quad (10)$$

This means that the geometric gauge scalar  $\varphi$  is to be understood as an additional gravitational potential. In this

regard scale invariance is built into our scheme in such a way that, in addition to the four degrees of freedom to make spacetime diffeomorphisms, a new gauge degree of freedom to make scale transformations arises (see the related discussion in sections III and VI). This intrinsic property of the scale invariant approaches which are associated with Weyl geometry has not been adequately discussed in former works where similar scenarios have been investigated [4, 10–12].

The model we are about to explore is not new (see for instance [4, 10]), however, as long as we know, its cosmological consequences, as well as its impact on particle physics phenomenology, have not been discussed in details. As we shall show, thanks to the adoption of Weyl-integrable geometry as the geometrical arena for the present setup, nothing besides scale invariance and cosmological inflation is necessary to explain the large hierarchy between the Planck and EW energy scales. Besides, due to the gauge freedom associated with scale invariance, the late-time speedup of the cosmic expansion can be explained without resorting to the exotic dark energy component of the cosmic budget. Moreover, the evidence on accelerated expansion can be explained in our setup even if the Universe is not expanding at all.

The paper has been organized in the following way. The details of the scale invariant theory of gravity and SMP which is the target of the present study are exposed in section II. This includes a discussion on the conservation of energy in our framework in subsection II C. In the section III particular attention is paid to the discussion of the gauge freedom arising in this theory as a consequence of scale invariance, which leads to a picture of our world which shares certain resemblance with the multiverse scenario [23]. One of the main features of our setup is discussed in section IV: varying masses of elementary particles and of composite systems. Several subtleties associated with the geodesics in Weyl spaces are also discussed in this section. The very important issue of identifying which quantities are physically meaningful and which ones are measured in experiments, is discussed in section V. This issue is fundamental to understand the way the large hierarchy between the Planck and EW energy scales arises in our setup. In order to illustrate the gauge freedom associated with scale invariance – previously discussed in section III – simple cosmological “solutions” to the Weyl-Einstein’s equations are derived in section VI. These are not solutions in the usual sense since, thanks to Weyl invariance, plain relationships between the cosmological scale factor and the Weyl gauge scalar are enough to satisfy the field equations while the cosmic dynamics remains unspecified. In section VII the origin of the redshift of frequencies is explained. This issue is important to understand how the late-time speedup of the cosmic expansion can be explained in this setup. The late-time acceleration of the expansion and the hierarchy between the Planck and EW energy scales are discussed in sections VIII and IX respectively. In this latter section the gauge freedom is analyzed again to point out that the

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<sup>5</sup> We point out that constants such as the bare masses of elementary particles, the Planck mass squared  $M_{\text{pl}}^2$ , and the cosmological constant  $\Lambda$  – among others – are not transformed by (1) plus  $\varphi \rightarrow \varphi + 2 \ln \Omega$ . The scale transformations we are considering here act only on field-dependent quantities and operators. For a related discussion see section II.

resolution of the mass hierarchy is gauge-dependent. In section X the singularity issue is investigated in connection with scale invariance, while section XI is dedicated to critically discuss the main differences of the present theory with a kind of generic scale-invariant formulations of particle physics and gravity which are designed to allow the construction of geodesically complete spacetimes [18, 20–22]. Physical discussion of the results and brief conclusions are given in section XII. In order for this paper to be self-contained we have added an appendix section XIII, where the fundamentals of Weyl-integrable geometry are exposed (XIII A). Besides, a demonstration of the local scale invariance of the EW Lagrangian is also included in this appendix (XIII B).

In this paper, following widespread conventions [1] – and for sake of convenience – we assume that the speed of light  $c$ , the Planck’s constant  $\hbar$ , and the electric charge of the electron  $q$ , behave as actual constants so that these are not transformed under the conformal transformation in (1) plus  $\varphi \rightarrow \varphi - 2 \ln \Omega$  (see the discussion on this issue in the next section). Unlike the approach of [12] (see also [18, 20–22]), in our setup, in order to avoid any disagreement with the existing observational constraints, we have removed the non-minimal coupling of the Higgs isodoublet to the curvature scalar (see Eq. (13)).

## II. SCALE INVARIANT THEORY OF GRAVITY MINIMALLY COUPLED TO THE SMP

In this section we expose the main features of the setup which, as we shall show, is capable of reconciling local scale invariance with EW broken symmetry (see also [4, 10, 11, 13]). The scale invariant theory of gravity minimally coupled to the SMP we will explore is associated with WIG spacetimes. Weyl-integrable geometry is obtained from the more general Weyl theory if make the replacement  $w_\mu \rightarrow \partial_\mu \varphi$ , where  $\varphi$  is known as the Weyl gauge scalar. In this case, since  $\oint dx^\mu \partial_\mu \varphi / 2 = 0$ , then the lengths of vectors, although point-dependent, are integrable. The torsionless affine connection of the WIG manifold – Eq. (3) with the replacement  $w_\mu \rightarrow \partial_\mu \varphi$  – is given by (10). From this point on, unless the contrary is specified, all geometric quantities and operators labeled with the “(w)” refer to WIG objects which are defined with respect to the affine connection (10). For a concise exposition of the fundamentals of Weyl-integrable geometry we submit the reader to the appendix XIII A.

The minimal gravitational action associated with WIG backgrounds which is invariant under the local scale transformations<sup>6</sup>

$$g_{\mu\nu} \rightarrow \Omega^{-2} g_{\mu\nu}, \quad \varphi \rightarrow \varphi + 2 \ln \Omega. \quad (11)$$

is the one in Eq. (9) (see also [4, 9–11]). Since in (9) the effective Planck mass  $M_{\text{pl}}^2(\varphi) = M_{\text{pl}}^2 e^\varphi$  is a point-dependent quantity, then, assuming  $M_{\text{pl}}(\varphi)$  to be the standard unit of mass, any mass parameter should share the same property:  $v_0^2(\varphi) = v_0^2 e^\varphi$ . The above is a direct consequence of adopting WIG backgrounds as the geometrical arena for the gravitational phenomena. In order to couple the SMP to the above scale invariant gravitational action, the terms within the action (5) associated with the Higgs field, with the appropriate replacements  $M_{\text{pl}}^2 \rightarrow M_{\text{pl}}^2(\varphi)$ ,  $v_0^2 \rightarrow v_0^2(\varphi)$ , should be added:

$$S^{(w)} = \int d^4x \sqrt{|g|} \left[ \frac{M_{\text{pl}}^2(\varphi) + \xi |h|^2}{2} R^{(w)} - \frac{1}{2} |Dh|^2 - \frac{\lambda}{4} (|h|^2 - v_0^2(\varphi))^2 \right],$$

where, as before,  $|Dh|^2 \equiv g^{\mu\nu} (D_\mu h)^\dagger (D_\nu h)$ , the gauge covariant derivative of the Higgs field is defined as  $D_\mu h = (D_\mu^* - \partial_\mu \varphi / 2) h$ , and

$$M_{\text{pl}}^2(\varphi) \equiv M_{\text{pl}}^2 e^\varphi, \quad v_0^2(\varphi) \equiv v_0^2 e^\varphi. \quad (12)$$

We point out that the non-minimal coupling of the Higgs field to the WIG curvature scalar is not relevant for the present theory. Hence, for simplicity, in what follows in this paper we remove this irrelevant coupling and set  $\xi = 0$ . As a consequence of this the setup we shall explore is governed by the simpler action:

$$S^{(w)} = \int d^4x \sqrt{|g|} \left[ \frac{M_{\text{pl}}^2(\varphi)}{2} R^{(w)} - \frac{1}{2} |Dh|^2 - \frac{\lambda}{4} (|h|^2 - v_0^2(\varphi))^2 \right], \quad (13)$$

where  $M_{\text{pl}}^2(\varphi)$  and  $v_0^2(\varphi)$  have been defined in (12). Also for simplicity of the analysis in (13) we have omitted the EW Lagrangian terms but for the Higgs boson. However, as it was clearly shown in [12], the EW terms missing in (13) do not spoil the scale invariance in Weylian backgrounds as long as the SMP fields are minimally coupled to gravity.<sup>7</sup> Since under the Weyl rescalings (11):

$$M_{\text{pl}}(\varphi) \rightarrow \Omega M_{\text{pl}}(\varphi), \quad v_0(\varphi) \rightarrow \Omega v_0(\varphi),$$

it is a simple exercise to show that the action (13) is invariant under (11) plus the following rescaling of the Higgs field:

<sup>6</sup> Here we will use the terminology “scale invariance” and “gauge invariance” interchangeably to mean invariance under the Weyl rescalings (11).

<sup>7</sup> For completeness in the appendix XIII B we have included the demonstration which was given in [12] step by step.

$$h \rightarrow \Omega h \Rightarrow D_\mu h \rightarrow \Omega D_\mu h. \quad (14)$$

Sometimes we shall call the transformations (11), (14), just as “scale or gauge transformations.”

Given that fundamental dimensionful constants such as the Plack mass are included in the action (13) from the start, the present theory is to be considered as less fundamental than the ones in [9, 11, 12, 18, 20–22]. In particular, the emergence of fundamental scales can not be addressed within our setup. Besides, the action (13) differs from the one in [18, 20–22] (see also [24–26]), in that the underlying geometric structure is WIG, and  $\varphi$  is no longer another singlet scalar field but it is just the Weyl gauge field of WIG geometry, i. e., the  $\varphi$ -kinetic energy term is already included in the WIG curvature scalar:

$$R^{(w)} = R - \frac{3}{2}(\partial\varphi)^2 - 3\Box\varphi, \quad (15)$$

where in the right-hand-side (RHS) of this equation the given quantities and operators coincide with their Riemannian definitions in terms of the Christoffel symbols of the metric,  $(\partial\varphi)^2 \equiv g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$ , and  $\Box\varphi \equiv g^{\mu\nu}\nabla_\mu\partial_\nu\varphi$ .

A detailed discussion on the differences of our setup with the generic Weyl-invariant theories of [18, 20–22] is presented in section XI.

### A. Varying constants

At this point we want to criticize a statement frequently found in the bibliography on scale invariance (see, for instance, Ref. [21]): “Weyl symmetry does not allow any dimensionful parameters in the action.” Such a statement were correct if implicitly assume Riemann geometry to govern the affine properties of spacetime and/or if the dimensionful parameters were not multiplied by appropriate powers of fields, i. e., if these parameters were not supposed to vary from point to point in spacetime, but not in general. The actual constants are not transformed by the Weyl transformations (11). Their existence may be taken as a convenient or unavoidable postulate if desired, but the fact is that every theory claimed to enjoy scale invariance has to deal with these truly constant quantities. However, if the given dimensionful constant is multiplied by an appropriate power of some field, take as an example the varying Planck mass squared:  $M_{\text{pl}}^2(\phi) = M_{\text{pl}}^2\phi^2$ , where under (11),  $\phi \rightarrow \Omega\phi$ , i. e.,  $M_{\text{pl}}^2(\phi) \rightarrow \Omega^2 M_{\text{pl}}^2(\phi)$ , then the resulting point-dependent Plack mass squared does actually transform under the Weyl rescalings and Weyl invariance might be preserved if consider terms like  $M_{\text{pl}}^2(\phi) [R + 6(\partial\phi)^2]$ , or  $M_{\text{pl}}^4(\phi)$ , in the action. The price to pay for allowing point-dependent fundamental constants in the scale-invariant action from the start is to renounce to “natural” generation of those fundamental constants by means

of symmetry breaking arguments. In this regard Weyl-invariant theories which do not include dimensionful constants in the action (2), (4), (5), and (8), are more fundamental than those which include point-dependent constants, such as the one of our setup which is depicted by the action (13).

In the present paper sometimes we shall call the actual constants like  $\hbar$ ,  $c = 1$ ,  $M_{\text{pl}}$ ,  $v_0$ , etc., as “bare” constants, in contrast to point-dependent “constants” which are obtained as a combination of a bare constant and some power of the Weyl gauge scalar. In the action (13) we have included the point-dependent Planck mass and EW mass parameter, respectively:  $M_{\text{pl}}^2(\varphi) = M_{\text{pl}}^2 e^\varphi$ ,  $v_0^2(\varphi) = v_0^2 e^\varphi$ , where the constants carry appropriate units while the gauge scalar  $\varphi$  is dimensionless. These point-dependent “constants” arise naturally in spacetimes whose affine geometrical structure is governed by Weyl geometry or any other modification of Riemann geometry which allows the lengths of vectors – these include the units of measure – to vary from point to point in spacetime.

### B. Field Equations

The WIG-Einstein field equations derived from (13) read:

$$G_{\mu\nu}^{(w)} = \frac{1}{M_{\text{pl}}^2(\varphi)} T_{\mu\nu}^{(m)} = \frac{e^{-\varphi}}{M_{\text{pl}}^2} T_{\mu\nu}^{(m)},$$

where  $G_{\mu\nu}^{(w)} \equiv R_{\mu\nu}^{(w)} - g_{\mu\nu}R^{(w)}/2$ , is the WIG-Einstein’s tensor and

$$T_{\mu\nu}^{(m)} = -\frac{2}{\sqrt{|g|}} \frac{\partial(\sqrt{|g|}\mathcal{L}_{\text{mat}})}{\partial g^{\mu\nu}}, \quad (16)$$

is the standard stress-energy tensor (SET) of the matter degrees of freedom. For the particular case of the Higgs field  $h$  in (13), the matter Lagrangian reads

$$\mathcal{L}_{\text{mat}} = -\frac{1}{2}|Dh|^2 - \frac{\lambda}{4}(|h|^2 - v_0^2 e^\varphi)^2. \quad (17)$$

Since the physics in the present theory must be expressed in terms of Weyl gauge covariant and/or invariant quantities – which include but are not limited to geometric invariants (see section X) – physically meaningful tensors of the same  $(n, m)$ -type or valence must transform in the same way under the Weyl rescalings (11). Take for instance the WIG-Ricci tensor  $R_{\mu\nu}^{(w)}$  or the related WIG-Einstein’s tensor  $G_{\mu\nu}^{(w)}$  which appears in the WIG-Einstein field equations above. These are not transformed by (11). Given that under the latter scale transformations,  $T_{\mu\nu}^{(m)} \rightarrow \Omega^2 T_{\mu\nu}^{(m)}$ , then the physically meaningful matter SET in the WIG-Einstein’s equations is given by:

$$T_{\mu\nu}^{(m,w)} \equiv e^{-\varphi} T_{\mu\nu}^{(m)}. \quad (18)$$

The latter is called as WIG-SET and, just like  $G_{\mu\nu}^{(w)}$ , it is not transformed by (11). I. e., it is not only gauge covariant but also scale invariant. The WIG-Einstein field equations can then be written in a manifestly gauge invariant form:

$$\begin{aligned} G_{\mu\nu}^{(w)} &= \frac{1}{M_{\text{pl}}^2} T_{\mu\nu}^{(w,m)} \Leftrightarrow \\ G_{\mu\nu} - \nabla_\mu \partial_\nu \varphi + g_{\mu\nu} \square \varphi + \\ &\frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi + \frac{1}{4} g_{\mu\nu} (\partial \varphi)^2 = \frac{e^{-\varphi}}{M_{\text{pl}}^2} T_{\mu\nu}^{(m)}, \end{aligned} \quad (19)$$

where in the second and third lines above the quantities and operators coincide with their Riemannian definitions in terms of the standard Christoffel symbols of the metric.

The Klein-Gordon (KG) equation which can be derived from (13) by taking variations with respect to  $\varphi$  which vanish on the boundary, is not an independent equation but it is just the trace of the Einstein-Weyl equations:

$$\begin{aligned} -R^{(w)} &= \frac{1}{M_{\text{pl}}^2} T^{(w,m)} \Leftrightarrow \\ \square \varphi + \frac{1}{2} (\partial \varphi)^2 - \frac{1}{3} R &= \frac{e^{-\varphi}}{3M_{\text{pl}}^2} T^{(m)}, \end{aligned} \quad (20)$$

where  $T^{(m)} = g^{\mu\nu} T_{\mu\nu}^{(m)}$  is the trace of the matter SET, i. e., the KG equation for the Weyl gauge field is a redundant equation.

In order to derive the KG equation for the Higgs field  $h$  in (13), it is recommended to redefine the Weyl gauge boson and to introduce the scale invariant Higgs scalar according to:

$$\phi = e^{\varphi/2}, \quad \chi = e^{-\varphi/2} h. \quad (21)$$

After this choice the Higgs piece of action in (13) reads

$$\begin{aligned} S_{\text{Higgs}} &= \int d^4x \sqrt{|g|} \phi^2 \mathcal{L}_{\text{Higgs}}, \\ \mathcal{L}_{\text{Higgs}} &= -\frac{1}{2} (\partial \chi)^2 - \frac{\lambda \phi^2}{4} (\chi^2 - v_0^2)^2. \end{aligned} \quad (22)$$

In terms of the new variables the WIG-Einstein field equations derived from (13) are

$$G_{\mu\nu}^{(w)} = \frac{1}{M_{\text{pl}}^2} \left[ T_{\mu\nu}^{(w,m)} + T_{\mu\nu}^{(w,\chi)} \right], \quad (23)$$

where, this time,  $T_{\mu\nu}^{(w,m)}$  is the gauge invariant SET for the matter degrees of freedom other than the Higgs field,

while the physically meaningful Higgs SET – in terms of the redefined fields  $\phi$ ,  $\chi$  – is given by the following expression:

$$\begin{aligned} T_{\mu\nu}^{(w,\chi)} &= \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} g_{\mu\nu} (\partial \chi)^2 \\ &\quad - \frac{\lambda \phi^2}{4} g_{\mu\nu} (\chi^2 - v_0^2)^2. \end{aligned} \quad (24)$$

Of course,  $T_{\mu\nu}^{(w,\chi)}$ , is also a gauge invariant quantity just like  $G_{\mu\nu}^{(w)}$  and  $T_{\mu\nu}^{(w,m)}$ . The WIG-Einstein field equation (23) is manifestly gauge invariant.

Written through only Riemannian quantities and operators the WIG-Einstein field equations (23) plus the KG equations for  $\phi$  and  $\chi$  read:

$$\begin{aligned} G_{\mu\nu} - \frac{2}{\phi} (\nabla_\mu \partial_\nu \phi - g_{\mu\nu} \square \phi) + 4 \frac{\partial_\mu \phi}{\phi} \frac{\partial_\nu \phi}{\phi} - \\ g_{\mu\nu} \frac{(\partial \phi)^2}{\phi^2} = \frac{1}{M_{\text{pl}}^2} \left[ \phi^{-2} T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(w,\chi)} \right], \end{aligned} \quad (25)$$

$$\begin{aligned} \square \phi - \frac{1}{6} R \phi = \frac{\phi}{6M_{\text{pl}}^2} \left[ \phi^{-2} T^{(m)} - (\partial \chi)^2 \right. \\ \left. - \lambda \phi^2 (\chi^2 - v_0^2)^2 \right], \end{aligned} \quad (26)$$

$$\square \chi + 2 \frac{(\partial \phi \cdot \partial \chi)}{\phi} = \lambda \phi^2 (\chi^2 - v_0^2) \chi, \quad (27)$$

where  $(\partial \phi \cdot \partial \chi) \equiv g^{\mu\nu} \partial_\mu \phi \partial_\nu \chi$ . We have to recall that  $T_{\mu\nu}^{(w,\chi)}$  in (25) is given by Eq. (24), and that (26) is not an independent equation since it coincides with the trace of Eq. (25). Recall also that  $\phi = e^{\varphi/2}$  is just the geometric Weyl gauge scalar which participates in the definition of the affine connection of the WIG manifold (10) with the replacement  $\varphi \rightarrow 2 \ln \phi$ .

### C. Conservation of energy

An issue which can not be avoided when dealing with scale invariance, is related with conservation of energy when matter fields are included in the theory (13). Let us consider first standard GR theory:

$$S_{\text{GR}} = \int d^4x \sqrt{|g|} \left[ \frac{R}{16\pi G} + \mathcal{L}_{\text{mat}} \right], \quad (28)$$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}^{(m)}, \quad (29)$$

where all quantities are Riemannian in nature, i. e., (28) deals exclusively with pseudo-Riemann spacetimes.

Due to the (contracted) Bianchi identity of the curvature tensor:

$$\nabla^\nu G_{\mu\nu} = 0 \Rightarrow \nabla^\nu T_{\nu\mu}^{(m)} = 0,$$

i. e., the SET of matter  $T_{\mu\nu}^{(m)}$  in Eq. (16) obeys the standard continuity equation. Under the conformal transformation of the metric in (11) the latter conservation equation is transformed into

$$\nabla^\nu T_{\nu\mu}^{(m)} = -T^{(m)} \partial_\mu \ln \Omega,$$

where  $T^{(m)} = g^{\mu\nu} T_{\mu\nu}^{(m)}$  is the trace of the SET. Hence, unless  $T^{(m)}$  vanishes – as it is for massless particles –, the non-vanishing trace spoils any existing scale invariance.

The above argument is true only if one deals with (pseudo)Riemann spaces. When WIG spaces are involved instead this argument is wrong. In fact, in this case, the geometric objects, including the WIG covariant derivative operator, are defined in terms of the affine connection of the Weyl-integrable geometry (10). Since, in the presence of matter with WIG-SET  $T_{\mu\nu}^{(w,m)}$  given by (18), (16), the WIG-Einstein equations derived from (13) are given by Eq. (19), then, the (contracted) Bianchi identity in WIG spacetimes

$$\nabla_{(w)}^\nu G_{\nu\mu}^{(w)} = 0, \quad (30)$$

entails the following manifestly gauge invariant conservation equation:

$$\nabla_{(w)}^\nu T_{\nu\mu}^{(w,m)} = 0 \Rightarrow \nabla_{(w)}^{(w)} T_{(w,m)}^{\nu\mu} = 2\partial_\nu \varphi T_{(w,m)}^{\nu\mu}, \quad (31)$$

where the “source” term in the RHS equation (31) expresses the fact that the units of measure of the stresses and energy are point-dependent units in WIG backgrounds, and under any circumstances it should be associated with any additional 5-force (compare with the geodesic equation (37)).

As seen it is the gauge invariant SET  $T_{\nu\mu}^{(w,m)}$  – and not  $T_{\mu\nu}^{(m)}$  – the one which is conserved in WIG spacetimes. This is expected since the scale invariant  $T_{\nu\mu}^{(w,m)}$  is the one with carries the physical meaning.

### III. GAUGE FREEDOM

As already mentioned, the KG equation for the Weyl gauge boson  $\varphi$  ( $\phi$  in the new variables) is not an independent equation, but it coincides with the trace of the WIG-Einstein equations (19) [Eq. (23) in the new variables which is rewritten in terms of Riemannian quantities in (25)]. This is a direct consequence of scale invariance which means that there is not just a single Weyl-integrable space  $(\mathcal{M}, g_{\mu\nu}, \varphi)$  which is solution of (25), (26) and (27), but a whole equivalence class of them:

$$\mathcal{C} = \{(\mathcal{M}, g_{\mu\nu}, \varphi) : \nabla_{(w)}^{(w)} g_{\alpha\beta} = -\partial_\mu \varphi g_{\alpha\beta}\}, \quad (32)$$

such that any other pair  $(\bar{g}_{\mu\nu}, \bar{\varphi})$  related with  $(g_{\mu\nu}, \varphi)$  by a scale transformation (11);

$$\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \bar{\varphi} = \varphi - 2 \ln \Omega, \quad (33)$$

also belongs in the conformal equivalence class  $\mathcal{C}$ . As we shall see in the cosmological examples in section VI, the most one can get from the field equations is a functional relationship among the gravitational potentials  $g_{\mu\nu}$  and  $\varphi$ . This relationship is independent of the matter content: for a given matter source one have an infinity of possibilities  $(g_{\mu\nu}, \varphi)$ . Each possible gauge  $(g_{\mu\nu}^a, \varphi_a) \in \mathcal{C}$ , where  $a = 1, 2, \dots, \infty$  and the  $\varphi_a$  belong in the space of continuous real-valued functions, represents a potential geometric description of the laws of gravity (and particle’s physics). From the cosmological standpoint, for instance, to have an infinity of feasible – fully equivalent – geometrical descriptions amounts to have an infinity of possible patterns of cosmological evolution which satisfy the cosmological field equations.

What does actually mean that the field equations (25), (26), and (27) are not enough to pick one such specific pair  $(g_{\mu\nu}, \varphi) \in \mathcal{C}$ ? What this means is that the field variables themselves have no independent physical meaning. Only physical reality – whatever it is – and gauge invariant quantities have independent physical meaning (see section V for a more detailed and elaborated discussion on this issue). Then, since in principle one is free to choose any given pair  $(g_{\mu\nu}, \varphi)$  as long as the field equations (25), (26) and (27) are satisfied, one might wonder, which is the meaning of such a (seemingly) inscrutable theory which is not capable of producing definite answers to given questions? To answer this question let us consider particle physics and cosmological arguments. Even if the field equations of our setup are not enough to pick one specific gauge, there are certain clues which help us to reduce the number of feasible possibilities. For instance, there is a problem with the hierarchy of mass scales. If one expects the theory (13) to account for the resolution of this problem, given that according to the field equations the following relationship between the cosmological scale factor and the Weyl gauge scalar  $\varphi$  arises (see section VI):  $e^{\varphi/2} \propto 1/a$ , and since the masses of any particles are point-dependent:  $m_p(\varphi) \propto e^{\varphi/2} \propto 1/a$  (see the next section), then, in order to have masses of the order of 1 TeV at the time of the EW phase transition – starting from Plack scale masses  $\propto 10^{19}$  GeV – one needs a prior stage of inflationary expansion. This fact alone rules out, for instance, the general relativity gauge – see below – where the mass hierarchy can not be even addressed. Other gauges can be also ruled out (see section IX A). As a matter of fact, not only the resolution of the mass hierarchy problem, but also of other open problems in particle physics and in cosmology, is gauge-dependent. This is, perhaps, the main point of our present discussion.

Our viewpoint is that, assuming that the present scale invariant theory is the one which governs the laws of gravity (at least at the classical level), the correct description of the cosmic dynamics of our Universe can be associated with one specific gauge, say,  $\varphi_g = \varphi_g(x)$ , which amounts

to pick one specific geometric description of our world, i. e., a specific pattern of cosmological evolution,  $(g_{\mu\nu}^g, \varphi_g)$ , among the infinity of equivalent geometrical pictures in  $\mathcal{C}$ . The cosmological observations help us to pick such a specific gauge. Once this gauge is fixed by the observations/experimentation, one is able to give definite answers to specific cosmological questions and to make definite predictions of relevance for cosmology as well. The chosen gauge will amount to our particular geometric understanding of the laws of gravity and particle's physics.

To summarize our viewpoint: there exist infinitely many different gauges comprised in our scale invariant setup – infinite many different spacetimes accordingly – but our own existence, which means that we can perform experiments and do observations, picks one specific gauge: the one which allows a correct description of the existing amount of observational/experimental evidence. This picture shares certain resemblance with the multi-verse scenario [23].

### A. The GR gauge

The simplest gauge one may choose is the one where  $\varphi = \varphi_0$  (in what follows, without loss of generality, we set the irrelevant constant  $\varphi_0 = 0$ ). This trivial gauge corresponds to general relativity since, after the above choice, the action (13) transforms into the EH action minimally coupled to the SMP with no new physics beyond the standard model at low energies:

$$S_{\text{GR}} = \int d^4x \sqrt{|g|} \left[ \frac{R}{16\pi G} - \frac{|D^*h|^2}{2} - \frac{\lambda}{4} (|h|^2 - v_0^2)^2 \right], \quad (34)$$

where  $8\pi G = M_{\text{pl}}^{-2}$ , and all quantities and operators are Riemannian objects.

It is a very simple exercise to show that the spacetimes which solve the GR field equations belong in the conformal class  $\mathcal{C}$ . Actually, let us consider the WIG-Einstein-Hilbert action  $S_{\text{EH}}^{(w)}$  given in (9). Under a Weyl rescaling (11) with  $\Omega^2 = e^\varphi$ , the WIG affine connection (10) is transformed into the Christoffel symbols of the conformal metric:

$$\Gamma_{\alpha\beta}^\mu \rightarrow \{\Gamma_{\alpha\beta}^\mu\} \Rightarrow R_{\mu\nu}^{(w)} \rightarrow R_{\mu\nu}, \text{ etc.}$$

then

$$S_{\text{EH}}^{(w)} \rightarrow S_{\text{EH}} = \int d^4x \sqrt{|g|} \frac{M_{\text{pl}}^2}{2} R.$$

Through an inverse transformation (11) with  $\Omega^2 = e^{-\varphi}$ , each GR solution generates an infinite set of spacetimes back in  $\mathcal{C}$ . To see this let us start with a given

known GR solution  $(\mathcal{M}, g_{\mu\nu}^{\text{GR}})$ , and perform the conformal transformation of the GR metric:

$$g_{\mu\nu}^{\text{GR}} \rightarrow \Omega^{-2} g_{\mu\nu}, \quad \Omega^2 = e^{-\varphi}.$$

Under the above transformation, the Riemannian quantities and operators are mapped back into WIG objects which are defined in terms of the affine connection (10) of the Weyl-integrable geometry:<sup>8</sup>

$$\{\Gamma_{\alpha\beta}^\mu\} \rightarrow \Gamma_{\alpha\beta}^\mu \Rightarrow R_{\mu\nu} \rightarrow R_{\mu\nu}^{(w)}, \text{ etc.}$$

It is not difficult to realize that there exists an infinite countable set of smooth real-valued functions  $\varphi_a = f_a(x)$ , such that every pair

$$(g_{\mu\nu}^a, \varphi_a), \text{ with } g_{\mu\nu}^a = e^{-\varphi_a} g_{\mu\nu}^{\text{GR}}, \quad a = 1, 2, \dots, \infty,$$

belongs in  $\mathcal{C}$ . This means that each GR spacetime solution  $(\mathcal{M}, g_{\mu\nu}^{\text{GR}})$ , together with its infinite set of equivalent conformal representations

$$\{(\mathcal{M}, g_{\mu\nu}^a, \varphi_a) : g_{\mu\nu}^a = e^{-\varphi_a} g_{\mu\nu}^{\text{GR}}, \quad a = 1, 2, \dots, \infty\},$$

belong in the equivalence class  $\mathcal{C}$ .

Unfortunately, general relativity is not able to correctly describe the observed pattern of cosmological evolution unless some exotic and mysterious “dark energy” is included in the cosmic budget. Hence, the GR-gauge does not seem to provide the correct description of our present Universe. The discussion on gauge freedom will be further illustrated in section VI within the cosmological setting.

## IV. VARYING MASSES AND GEODESICS

In the gravitational theory (13) gravity is propagated both by the metric and by the gauge Weyl scalar, so that this is a scalar-tensor theory. However, unlike other scalar-tensor theories like, for instance, Brans-Dicke (BD) gravity [27], since both  $g_{\mu\nu}$  and  $\varphi$  contribute towards the curvature of spacetime, in the present theory gravity is a fully geometrical phenomenon.<sup>9</sup> In consequence, since the mass of bosons and fermions of the SMP are related with the point-dependent symmetry breaking mass parameter  $v_0(\varphi)$  in Eq. (12), the masses of the elementary particles are influenced by the spacetime curvature through the gauge scalar  $\varphi$ :

$$m_p(\varphi) \propto g_p v_0(\varphi) = g_p v_0 e^{\varphi/2},$$

<sup>8</sup> Another possible reading of this (inverse) conformal transformation – as a matter of fact, the most popular one – will be discussed in section XI.

<sup>9</sup> It is very simple to see that the gauge scalar  $\varphi$  contributes towards the curvature of spacetime. Actually, take as an example the WIG-curvature scalar in Eq. (15). Even if the spacetime metric is flat, say  $g_{\mu\nu} = \eta_{\mu\nu}$ , the WIG curvature scalar is non-vanishing:  $R^{(w)} = -3\Box\varphi - 3(\partial\varphi)^2/2$ .



where  $g_p$  is some gauge coupling. In this paper we adopt that not only the mass of elementary particles, but also the masses of composite systems like hadrons, atoms, molecules, etc., and of macroscopic bodies, depend on spacetime point in the same way:

$$m(\varphi) = m_0 e^{\varphi/2}. \quad (35)$$

This assumption is consistent with experimental evidence on the equivalence principle [28] and on variation of electron-to-proton mass relation [29], among others.

Point-dependent masses entail that the 4-momentum  $p^\mu$  of a particle must be defined as

$$p^\mu := m(\varphi) \frac{dx^\mu}{d\tau} \Rightarrow g_{\mu\nu} p^\mu p^\nu = -m^2(\varphi), \quad (36)$$

where  $d\tau = -ids$ , and we are assuming the metric signature  $(-+++)$ . It follows that, since under (11):

$$d\tau \rightarrow \Omega^{-1} d\tau, \quad m(\varphi) \rightarrow \Omega m(\varphi),$$

then

$$p^\mu \rightarrow \Omega^2 p^\mu, \quad p_\mu \rightarrow p_\mu.$$

The geodesic equation of a particle with momentum  $p^\mu$  defined in (36), which is moving in a WIG spacetime, is given by [4]:

$$\frac{dp^\mu}{d\tau} + \Gamma^\mu_{\sigma\lambda} \frac{dx^\sigma}{d\tau} p^\lambda = \partial_\lambda \varphi \frac{dx^\lambda}{d\tau} p^\mu. \quad (37)$$

The term in the RHS of this equation is associated with the point-dependent property of the particle's mass in WIG spacetimes and has nothing to do with any additional 5-force (compare with the RHS equation (31)). This is evident if rewrite the Eq. (37) in the equivalent form:

$$\frac{d^2 x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\sigma} \frac{dx^\lambda}{d\sigma} = 0, \quad (38)$$

where we have conveniently rescaled the affine parameter  $d\sigma = e^{\varphi/2} d\tau$ . In the section V we shall explain why the latter gauge invariant affine parameter  $\sigma = \int \exp(\varphi/2) d\tau$  is not the proper time measured by an observer at rest.

Given that the electromagnetic 4-vector  $A_\mu$  and  $p_\mu$  transform in the same fashion under the Weyl transformations – in fact these are not transformed under (11), i. e., these are gauge invariant quantities – the above definition of the four-momentum in Eq. (36) admits a gauge extension which is consistent with scale invariance:

$$p_\mu \rightarrow p_\mu + q A_\mu \Leftrightarrow \partial_\mu \rightarrow \partial_\mu + iq A_\mu.$$

If instead of the varying mass  $m(\varphi) = m_0 e^{\varphi/2}$ , in the definition of  $p^\mu$  – equation (36) – one considers the bare mass

$m_0$ , then, under (11):  $p_\mu \rightarrow \Omega^{-1} p_\mu$ , while  $A_\mu \rightarrow A_\mu$ , which means that the only way around for gauge extension compatible with scale invariance is to admit varying electric charge  $q(\varphi) = e^{-\varphi/2} q$ , a possibility which is excluded in this work.

Another argument in favor of the definition (36) of the 4-momentum can be explained as follows. Let us consider in Eq. (37) a Lorentz-force term

$$f^\mu \equiv q F^\mu_{\lambda} \frac{dx^\lambda}{d\tau}, \quad (39)$$

where  $q$  is an electric charge and  $F_{\mu\nu}$  is the electromagnetic strength tensor. Hence, the particle with 4-momentum  $p^\mu$  and electric charge  $q$  obeys the following equation of motion:

$$\frac{dp^\mu}{d\tau} + \Gamma^\mu_{\sigma\lambda} \frac{dx^\sigma}{d\tau} p^\lambda - \partial_\lambda \varphi \frac{dx^\lambda}{d\tau} p^\mu = f^\mu. \quad (40)$$

Now, under the scale transformations (11) the Lorentz force (39) transforms like:

$$f^\mu \rightarrow \Omega^3 f^\mu.$$

It is a very instructive exercise to show that the only possible definition of the 4-momentum which allows scale invariance of Eq. (40) is the one in (36). Actually, taking into account (36), under (11) the left-hand-side of (40) transforms in the following way:

$$\begin{aligned} \frac{dp^\mu}{d\tau} + \Gamma^\mu_{\sigma\lambda} \frac{dx^\sigma}{d\tau} p^\lambda - \partial_\lambda \varphi \frac{dx^\lambda}{d\tau} p^\mu \rightarrow \\ \Omega^3 \left[ \frac{dp^\mu}{d\tau} + \Gamma^\mu_{\sigma\lambda} \frac{dx^\sigma}{d\tau} p^\lambda - \partial_\lambda \varphi \frac{dx^\lambda}{d\tau} p^\mu \right]. \end{aligned} \quad (41)$$

As a counter-example, the standard GR definition  $\hat{p}^\mu = m_0(dx^\mu/d\tau)$  explicitly breaks the scale invariance of (40), and also of (37), since, in the mentioned case, given that under (11)  $\hat{p}^\mu \rightarrow \Omega \hat{p}^\mu$ , then

$$\begin{aligned} \frac{d\hat{p}^\mu}{d\tau} + \Gamma^\mu_{\sigma\lambda} \frac{dx^\sigma}{d\tau} \hat{p}^\lambda - \partial_\lambda \varphi \frac{dx^\lambda}{d\tau} \hat{p}^\mu \rightarrow \Omega^2 \left[ \frac{d\hat{p}^\mu}{d\tau} \right. \\ \left. + \Gamma^\mu_{\sigma\lambda} \frac{dx^\sigma}{d\tau} \hat{p}^\lambda - \partial_\lambda \varphi \frac{dx^\lambda}{d\tau} \hat{p}^\mu - \partial_\lambda (\ln \Omega) \frac{dx^\lambda}{d\tau} \hat{p}^\mu \right]. \end{aligned}$$

Another interesting possibility is suggested by the WIG geodesic equation in the form (38):

$$\bar{p}^\mu = m_0 \frac{dx^\mu}{d\sigma}, \quad (42)$$

where  $\bar{p}^\mu$  is gauge invariant, but then there is no natural way to couple the Lorentz force  $\bar{f}^\mu = q F^\mu_{\lambda} (dx^\lambda/d\sigma)$  to (38) in a gauge covariant manner, unless we write

$$\frac{d\bar{p}^\mu}{d\sigma} + \Gamma^\mu_{\kappa\lambda} \frac{dx^\kappa}{d\sigma} \bar{p}^\lambda = e^{-\varphi} \bar{f}^\mu,$$

which is very unappealing. Besides, since  $\bar{p}^\mu$  is invariant under (11), then the only way to write gauge covariant combinations like  $p_\mu + qA_\mu$  is in the following unappealing way:  $\bar{p}_\mu + qe^{-\varphi}A_\mu$ . This suggests that, perhaps, one might consider a point-dependent electric charge  $q(\varphi) = qe^{-\varphi}$ , a possibility which is excluded in the present investigation. This is in addition to the fact that in the definition (42) neither  $m_0$  - the bare rest mass, nor  $\sigma$  - the gauge invariant affine parameter along the geodesics in (38), are the quantities which are measured by co-moving WIG observers [for a related discussion on the possibility of considering  $\sigma$  as a measure of proper time see the next section V]. Hence, we agree that the only feasible way within the present setup is to adopt varying masses (35) and the corresponding definition of the 4-momentum given by Eq. (36).

### A. Null geodesics

For massless particles the geodesic equations look the same as (37) but with replacement of the 4-momentum by the wave vector

$$p^\mu \rightarrow k^\mu \equiv \frac{dx^\mu}{d\lambda}, \quad (43)$$

where  $\nu = dx^0/d\lambda$  is the photon's frequency, and  $\lambda$  is an affine parameter along the photon's path which, under (11), transforms like:<sup>10</sup>  $d\lambda \rightarrow \Omega^{-2}d\lambda$ . Notice that under (11),  $k^\mu \rightarrow \Omega^2 k^\mu$ , while  $k_\mu \rightarrow k_\mu$ . The null geodesics read:

$$\frac{dk^\mu}{d\lambda} + \Gamma^\mu_{\sigma\nu} k^\sigma k^\nu = \partial_\sigma \varphi k^\sigma k^\mu,$$

which, since  $g_{\mu\nu} k^\mu k^\nu = 0$ , can be rewritten in the simplified GR form:

$$\frac{dk^\mu}{d\lambda} + \{\mu_{\sigma\nu}\} k^\sigma k^\nu = 0, \quad (44)$$

where, as above,  $\{\mu_{\sigma\nu}\}$  are the standard Christoffel symbols of the metric. I. e., the standard Riemannian geodesic of a massless particle in general relativity is already scale invariant. This is not a surprise since the scale transformations (11) – which leave the speed of light unaltered – do not modify the causal structure of the space-time.

## V. GAUGE INVARIANTS AND MEASURED QUANTITIES

In this section the discussion goes on those quantities which have an independent gauge invariant physical

meaning and those which are actually measured in (or inferred from) experiments/observations. In a separate section X particular attention will be paid to the gauge independent curvature invariants in connection with the singularity issue. The more technical and formal – perhaps also philosophical – issue of which quantities are to be regarded as the observables of the theory, of great importance for the study of quantum aspects, is behind the scope of the present paper and will not be considered here.

It is almost obvious to every freshman that the invariants of a given theory and those quantities which are actually measured in physical experiments are not one and the same thing. Take, for instance, the theory of relativity. While an observer in the rest frame<sup>11</sup> measures the rest mass,  $m_0$ , which is an actual relativistic invariant, another one in a moving frame measures the quantity,  $m = m_0/\sqrt{1-u^2}$ , where  $u$  is the relative speed of the moving observer with respect to the observer at rest with the mass.<sup>12</sup> Another very interesting illustration is provided by the experiment of Rossi and Hall [31], where the population of cosmic-ray-produced muons at the top of a mountain was compared to that observed at sea level. Although the travel-time for the muons from the top of the mountain to the base is several muon's lifetimes, the muon sample at the base was only moderately reduced. This can be explained by the time dilation attributed to their high relative speed:

$$\Delta t = \frac{\tau}{\sqrt{1-v^2}},$$

where  $\Delta t$  is the muon's travel-time in the experiment,  $\tau$  is the muon's lifetime which is measured in the rest frame, while  $v$  is the speed of the muons relative to the experimenters. Since  $\Delta t$  was measured as well as the relative speed, which in the experiment was  $v \approx 0.99$ , the lifetime of the muon was then inferred (computed). In Thompson's experiment, for instance, what one directly measures is the distance traveled by the cathode rays, their deflection and the electric and magnetic fields (see subsection VB below). The charge-to-mass ratio of the particles in the stream is then inferred.

In the above examples the measured quantities are  $m$  - the null-component of the 4-momentum, the speed-

<sup>11</sup> It is implicit that given observers are equipped with the necessary apparatus to make measurements.

<sup>12</sup> In general, given a particle of mass  $m_0$  with 4-momentum  $p^\mu = m_0 u^\mu$  ( $u^\mu = dx^\mu/d\tau$ ,  $d\tau = -ids \Rightarrow u_\mu u^\mu = -1$ ), the energy of the particle as measured by an observer whose 4-velocity is  $v^\mu$ , is defined by  $\mathcal{E} = -p_\mu v^\mu$  [30]. For a particle at rest with respect to the observer,  $v^\mu = u^\mu$ ,  $\mathcal{E} = -p_\mu v^\mu = m_0$ . For continuous matter distributions with SET  $T_{\mu\nu}^{(m)}$ , the quantity

$$\rho = T_{\mu\nu}^{(m)} v^\mu v^\nu,$$

is interpreted as the energy density as measured by an observer with 4-velocity  $v^\mu$ . It is seen that the measured energy density depends on the relative speed of the observer.

<sup>10</sup> See Eq. (D.6), pag. 446 in the appendix D of reference [30].

dependent time travel  $\Delta t$  (an interval of coordinate time), the relative speeds, and the components of  $\mathbf{E}$  - the electric, and of  $\mathbf{B}$  - the magnetic fields, i. e., the 6 components of the electromagnetic field strength  $F_{\mu\nu}$ , etc. The point is that sometimes what one measures in experiments are the components of tensors, besides these measurements depend on the reference frame. With the measured components of given tensors one can compute the related invariants of the theory:  $p_\mu p^\mu = -m_0^2$ ,  $\Delta x_\mu \Delta x^\mu = -\Delta\tau^2$ ,  $F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2)$ ,  $*F_{\mu\nu} F^{\mu\nu} = -4\mathbf{B} \cdot \mathbf{E}$ , etc., where  $*F_{\mu\nu}$  is the dual field strength. These invariants are the ones which carry actual physical meaning independent of the reference frame.

A similar situation occurs in our setup. Sometimes what one measures in (or infers from) experiments are the components of gauge covariant and/or invariant quantities. This is in contrast to the fact that meaningful physical conclusions are encoded in the gauge-independent covariant objects and invariants themselves. In the theory (13) such scale invariant quantities are, for instance, the WIG-Riemann-Christoffel curvature tensor  $R_{\mu\nu\kappa\lambda}^{(w)}$ , the corresponding WIG-Ricci tensor  $R_{\mu\nu}^{(w)}$  and WIG-Einstein's tensor  $G_{\mu\nu}^{(w)}$ , the gauge independent SET of matter  $T_{\mu\nu}^{(w,m)}$ , the 4-momentum  $p_\mu$ , the electromagnetic 4-vector  $A_\mu$ , the wave 4-vector  $k_\mu$ , etc. The gauge-independent curvature invariants

$$e^{-\varphi} R^{(w)}, e^{-2\varphi} R_{\mu\nu}^{(w)} R^{\mu\nu}_{(w)}, e^{-4\varphi} R_{\mu\nu\kappa\lambda}^{(w)} R^{\mu\nu\kappa\lambda}_{(w)}, \quad (45)$$

etc, which are the physically meaningful curvature invariants of the theory, will be discussed in section X in connection with the singularity issue.

Before going further we want to briefly comment on the affine parameter along the geodesics (see Eq. (38)):

$$\sigma = \int e^{\varphi/2} d\tau = -i \int e^{\varphi/2} ds.$$

Assuming that it is the proper time measured by an observer at rest in a WIG spacetime, as discussed in [10], one would have:

$$d\sigma^2 = -e^\varphi g_{\mu\nu} dx^\mu dx^\nu, \quad (46)$$

where the invariance under (11) is manifest. However, such a factorization of the metric is not unique; nothing stops us from doing coordinate transformations of the kind

$$dx^\mu \rightarrow e^{-\varphi/2} dx^\mu \Rightarrow d\sigma^2 = -g_{\mu\nu} dx^\mu dx^\nu,$$

where now the Weyl rescalings (11) are to be understood just as spacetime diffeomorphisms,

$$g_{\mu\nu} \rightarrow \Omega^{-2} g_{\mu\nu}, dx^\mu \rightarrow \Omega dx^\mu \Rightarrow d\sigma^2 \rightarrow d\sigma^2.$$

This is in contrast to our statement about (11) as acting on the fields only, i. e., from the start - in line

with Dicke's understanding of the transformations of units [1] - we excluded the possibility of considering the Weyl rescalings as spacetime diffeomorphisms. We conclude that the understanding of the conformal transformation in (11) as a transformation of units a la Dicke makes sense only if preserve the fundamental character of the standard line-element,  $ds = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$ , being a gauge covariant quantity which under (11) transforms like  $ds \rightarrow \Omega^{-1} ds$ . In this regard, the proper time measured by an observer at rest in the theory (13) is  $\tau = -i \int ds$ .

In the next subsections we shall discuss on the way the following two quantities which are cornerstone to understand how the large hierarchy between the Plack and EW scales arises in our setup (see section IX), are measured in experiments: (i) the gravitational coupling or Newton's constant (subsection V A), and (ii) the rest mass of charged particles (subsection V B).

### A. Measuring the Newton's constant

As already stressed (see section II B) the physically meaningful SET in the WIG-Einstein equations (19) is the gauge invariant SET of matter  $T_{\mu\nu}^{(w,m)}$  (18) which, as shown in the subsection II C, is the one conserved in WIG spacetimes. Take as an illustration a perfect fluid with stress-energy tensor

$$T_{\mu\nu}^{(w,m)} = [\rho^{(w)} + P^{(w)}] u_\mu u_\nu + P^{(w)} g_{\mu\nu}, \quad (47)$$

where  $u^\mu = dx^\mu/d\tau$  ( $d\tau = -ids$ ), and

$$\rho^{(w)} = e^{-\varphi} \rho, P^{(w)} = e^{-\varphi} P, P^{(w)} = (\gamma - 1)\rho^{(w)}, \quad (48)$$

are the WIG energy density and barotropic pressure respectively. The energy density measured by physical observers which are co-moving with the perfect fluid - their 4-velocity  $u^\mu$  coincides with that of the fluid itself - is

$$\rho^{(w)} = T_{\mu\nu}^{(w,m)} u^\mu u^\nu, \quad (49)$$

and not  $\rho$ . As a consequence the gravitational coupling measured in Cavendish-type experiments - see the related computations below - is just the usual Newton's constant  $G$ , a fact which is evident also from the WIG-Einstein's equations (19). Notice that under (11) the standard GR (Riemannian) energy density and the one which is measured by WIG observers transform differently:

$$\rho \rightarrow \Omega^4 \rho, \rho^{(w)} \rightarrow \Omega^2 \rho^{(w)}.$$

In the Newtonian limit where particles move slowly  $dx^i/ds \ll dt/ds$ , and only weak static gravitational field is considered, assuming small departure from Riemannian geometry  $\partial_i \varphi \sim dx^i/dt$ :

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \varphi = \varphi_0 + \phi, \\ |h_{\mu\nu}| \ll 1, \quad \phi \ll 1, \quad \frac{\partial h_{\mu\nu}}{\partial t} = 0, \quad \frac{\partial \phi}{\partial t} = 0,$$

we have for the geodesic motion:

$$\frac{d}{ds} \left( \frac{dx^\mu}{ds} \right) + \Gamma^\mu_{00} \left( \frac{dt}{ds} \right)^2 = 0,$$

so that

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2} \partial_i (h_{00} - \phi) \Rightarrow h_{00} - \phi = -2\Phi,$$

where  $\Phi$  is the Newtonian gravitational potential. In the same limit, since

$$T_{00}^{(w,m)} = \rho^{(w)} \gg |T_{ik}^{(w,m)}|, \quad T^{(w,m)} = -T_{00}^{(w,m)},$$

the WIG-Einstein field equations lead to ( $\nabla^2 \equiv \partial_i^2$ )

$$R_{00}^{(w)} = \frac{1}{2M_{\text{pl}}^2} T_{00}^{(w,m)} \Rightarrow \nabla^2 (h_{00} - \phi) = -\frac{\rho^{(w)}}{M_{\text{pl}}^2} \\ \Rightarrow \nabla^2 \Phi = \frac{\rho^{(w)}}{2M_{\text{pl}}^2} = 4\pi G \rho^{(w)},$$

so that, for a homogeneous distribution of matter confined within the volume  $\Omega_M$ :

$$\Phi = -G \frac{M}{r}, \quad M = \int_{\Omega_M} d^3x \rho^{(w)},$$

where it is apparent that it is the Newton's constant  $G = M_{\text{pl}}^{-2}/8\pi$ , and not the varying  $G(\varphi) = M_{\text{pl}}^{-2}(\varphi)/8\pi$ , the one measured in Cavendish experiments.

## B. Measuring the masses of particles

Imagine a particle with bare mass  $m_0$  and electric charge  $q$  which is moving with velocity  $\mathbf{v}$  in an electromagnetic field with field strength  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ . Assume a WIG spacetime background. Under these assumptions the motion of the particle is governed by equation (40):

$$\frac{dp^\mu}{d\tau} + \Gamma^\mu_{\sigma\lambda} \frac{dx^\sigma}{d\tau} p^\lambda - \partial_\lambda \varphi \frac{dx^\lambda}{d\tau} p^\mu = q F^\mu{}_\lambda \frac{dx^\lambda}{d\tau},$$

or, in terms of the Christoffel symbols of the metric

$$\frac{dp^\mu}{d\tau} + \{\mu_{\sigma\lambda}\} \frac{dx^\sigma}{d\tau} p^\lambda + \frac{m(\varphi)}{2} \partial^\mu \varphi = q F^\mu{}_\lambda \frac{dx^\lambda}{d\tau}, \quad (50)$$

where the 4-momentum of the particle is defined by (36). Let's consider further the Newtonian limit – see above – where particles move slowly  $dx^i/ds \ll dt/ds$ , departures

from Riemannian geometry are small  $\partial_i \varphi \sim dx^i/dt$ , and gravitation is weak and static  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ,  $\varphi = \varphi_0 + \phi$ . After these simplifying constraints the motion equations (50) can be written as it follows (here we adopt 3D vector notation):

$$\frac{d\mathbf{v}}{dt} = -\nabla\Phi + \frac{q}{m(\varphi)} (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}), \quad (51)$$

where  $\mathbf{E}$  and  $\mathbf{B}$  are the electric and magnetic fields respectively, and, as before,  $\Phi = (\phi - h_{00})/2$  - the Newtonian gravitational potential. The differences of (51) with the standard equation in general relativity are in the Newtonian potential which in GR is entirely due to the (0,0)-component of the metric tensor, and in the mass in the denominator of the second term in the RHS of (51). In the standard expression it is the bare mass  $m_0$  which appears instead of the point-dependent mass  $m(\varphi)$ .

Let us analyze the Thompson's experiment with cathode rays where the charge-to-mass ratio is computed in terms of measured quantities, on the light of Eq. (51). Suppose that the cathode rays – a stream of particles of mass  $m$  and electric charge  $q$  – are initially traveling with a speed  $v$  in the  $x^1$ -direction, subject to a uniform electric field  $E$  in the  $x^3$ -direction and a uniform magnetic field  $B$  in the  $-x^2$ -direction. Then, neglecting the gravitational effect, Eq. (51) can be written as:

$$\frac{d^2 x^3}{dt^2} = \frac{q}{m(\varphi)} (E - vB). \quad (52)$$

In Thompson's experiment the electric field is turned on first and the deflection  $d$  of the ray in the  $x^3$ -direction is measured after it had traveled a distance  $l \gg d$  in the electric field:

$$d = \frac{q}{m(\varphi)} \frac{El^2}{2v^2},$$

where we have replaced the time of flight  $t$  by  $l/v$ . Then the magnetic field is turned on, and the apparatus adjusted in such a way that the cathode ray is no longer deflected, meaning an exact balance between the electric and magnetic forces:  $E = vB$ . If substitute  $v$  from the latter equation back into the former expression, one can compute the charge-to-mass ratio:

$$\frac{q}{m(\varphi)} = \left( \frac{2d}{l^2} \right) \frac{E}{B^2}.$$

The point is that, in the above Thompson's experiment one measures  $d$ ,  $l$ ,  $E$  and  $B$ . If we are capable of measuring also the electric charge – say, in a different experiment like the Millikan's "oil drop" experiment – one can infer/compute the mass of the particle. But notice that it is not the bare mass  $m_0$  what one computes but the point-dependent one:  $m(\varphi) = m_0 e^{\varphi/2}$ . In local experiments involving weak gravitational fields  $e^{\varphi/2}$  can

be assumed an unimportant almost constant scale factor that can be taken away by an appropriate rescaling of constants. However, if cosmological effects are to be considered, or a very strong gravitational field is present, this factor can be very important. As a matter of fact, as we shall see in section IX, the mentioned factor and cosmological inflation may explain the large mass hierarchy between the Planck and EW scales.

## VI. COSMOLOGICAL SOLUTIONS

In order to illustrate the gauge freedom arising in the present setup due to scale invariance, in this section we shall look for simple cosmological solutions of the theory (13). We shall assume Friedmann-Robertson-Walker (FRW) spacetimes with flat spatial sections, which are depicted by the following line element:

$$ds^2 = -dt^2 + a^2(t)\delta_{ij} dx^i dx^j, \quad (53)$$

where  $t$  is the cosmic time and  $a(t)$  is the scale factor.

### A. Vacuum cosmology

As already discussed in section III, one very interesting property of the theory (13) is a direct consequence of scale invariance: There is a whole conformal equivalence class  $\mathcal{C}$  of spacetimes (see Eq. (32)) which amount to equivalent geometrical descriptions of a same phenomenon. Mathematically this means that, in addition to the four degrees of freedom to make spacetime diffeomorphisms, a new degree of freedom to make scale transformations arises. This property is not shared by none of the existing scale invariant theories of the SMP coupled to gravity [9, 11, 12, 16–18, 20–22], or at least, it has not been properly discussed before. To illustrate this gauge freedom let us consider vacuum cosmology within our setup (13) in a flat FRW background given by (53). The vacuum field equations which are derived from (13) [ $G_{\mu\nu}^{(w)} = 0$ ] can then be written as follows:

$$3 \left( H + \frac{1}{2} \dot{\varphi} \right)^2 = 0, \quad \dot{H} + \frac{1}{2} \ddot{\varphi} = 0, \quad (54)$$

where  $H \equiv \dot{a}/a$ , and the dot accounts for  $t$ -derivative. The Friedmann equation above can be integrated to obtain the following dependence of the scale factor upon the gauge field  $\varphi$ :

$$a(\varphi) = a_0 e^{-\varphi/2}, \quad a_0 = e^{C_0}, \quad (55)$$

where  $C_0$  is an arbitrary integration constant. If we substitute this  $a(\varphi)$  back into the remaining equations – second equation in (54) and KG equation – these become

just identities, so that no new information can be extracted from them. Hence the above is not a solution in the conventional sense since the cosmic dynamics remains unspecified. This is a consequence of scale invariance which gives us the freedom to choose either any  $\varphi(t)$ , or any  $a(t)$  we want. Recall that one of these degrees of freedom can be transformed in any desired way by an appropriate scale transformation of the kind (11).

In the case when a cosmological term is considered,

$$G_{\mu\nu}^{(w)} = -\Lambda e^\varphi g_{\mu\nu} \Rightarrow 3 \left( H + \frac{1}{2} \dot{\varphi} \right)^2 = \Lambda e^\varphi, \quad (56)$$

assuming de Sitter expansion

$$H = H_0 \Rightarrow a = a_0 e^{H_0 t},$$

the relationship between the scale factor and the gauge scalar is modified:

$$e^{\varphi/2} = \frac{1}{1 + a/a_0} = \frac{1}{e^{H_0 t} + 1}, \quad (57)$$

where, for simplicity, we have assumed that  $H_0 = \sqrt{\Lambda/3}$ . At small  $a \ll a_0$  ( $t \ll 1/H_0$ ),  $\varphi$  is almost a constant which corresponds to the GR gauge, while at large  $a \gg a_0$  ( $t \gg 1/H_0$ ), the relationship (55) is recovered.

### B. Matter fields

Let us assume that the WIG-Einstein equations (19) are sourced by matter in the form of a perfect fluid with WIG-SET defined in equations (47), (48). In this case the (0,0)-component of the WIG-Einstein equations (19) – properly the WIG-Friedmann equation – and the conservation equation (31) can be written as follows

$$3 \left( H + \frac{1}{2} \dot{\varphi} \right)^2 = \frac{1}{M_{\text{pl}}^2} \rho^{(w)},$$

$$\dot{\rho}^{(w)} + 3 \left( H + \frac{1}{2} \dot{\varphi} \right) [\rho^{(w)} + P^{(w)}] = \dot{\varphi} \rho^{(w)}. \quad (58)$$

When the cosmological fluid is just vacuum, we have that  $P^{(w)} = -\rho^{(w)}$ , so that the conservation equation in (58) simplifies  $\dot{\rho}^{(w)} = \dot{\varphi} \rho^{(w)}$ . This can be easily integrated:  $\rho^{(w)} = C e^\varphi$ , where  $C$  is a constant (compare with the cosmological term in Eq. (56)).

In general, when the following state equation is valid:  $P^{(w)} = (\gamma - 1)\rho^{(w)}$ , where  $\gamma$  is the barotropic parameter of the cosmological fluid, the conservation equation in (58) can be written as

$$\dot{\rho}^{(w)} + 3\gamma \left( H + \frac{1}{2} \dot{\varphi} \right) \rho^{(w)} = \dot{\varphi} \rho^{(w)},$$

which can be immediately integrated to yield the following dependence of the energy density on the gauge scalar and on the scale factor:

$$\rho^{(w)} = \rho_0 e^{-(3\gamma-2)\varphi/2} a^{-3\gamma}.$$

For radiation

$$\rho_{\text{rad}}^{(w)} = \rho_{0,r} e^{-\varphi} a^{-4}, \quad (59)$$

while, for dust

$$\rho_{\text{dust}}^{(w)} = \rho_{0,d} e^{-\varphi/2} a^{-3}. \quad (60)$$

Given the gauge freedom to make Weyl transformations, in order to obtain more information about the relationship between the gauge scalar and the scale factor from the Friedmann equation in (58), one needs to make some statement about the cosmic dynamics. In the case of the radiation fluid, we may assume that – as in standard GR – the scale factor evolves according to

$$a(t) = a_{0,r} \sqrt{t} \Rightarrow H = \frac{1}{2t} = \frac{1}{2} \left( \frac{a_{0,r}}{a} \right)^2, \quad (61)$$

where  $a_{0,r}$  is an arbitrary constant. Since, according to (61),  $a^2 H = a_{0,r}^2/2$ , then the Friedmann equation in (58) can be written as

$$\frac{d\varphi}{d\tau} = e^{-\varphi/2} - 1,$$

where we have set  $a_{0,r} = (4\rho_{0,r}/3M_{\text{pl}}^2)^{1/4}$  and a new variable  $\tau \equiv \ln a^2$  has been introduced. The solution of this equation is easily derived:

$$e^{\varphi/2} = 1 + \frac{a_*}{a} \Rightarrow \rho_{\text{rad}}^{(w)} = \frac{\rho_{0,r}}{a^4(1 + a_*/a)^2}, \quad (62)$$

where  $a_*$  is another arbitrary constant.

In the dust case, following GR results, one may impose that

$$a(t) = a_{0,d} t^{2/3} \Rightarrow a^{3/2} H = 2a_{0,d}^{3/2}/3, \quad (63)$$

where  $a_{0,d}$  is a constant. It follows that the Friedmann equation in (58) can be written as

$$\frac{d\varphi}{d\tau} = e^{-\varphi/4} - 1,$$

where, as before,  $\tau = \ln a^2$ , and we have set  $a_{0,d} = (4\rho_{0,d}/3M_{\text{pl}}^2)^{1/3}$ . Integration of this last equation leads to the following relationship between the gauge scalar and the scale factor:

$$e^{\varphi/2} = \left( 1 + \sqrt{\frac{a_{\dagger}}{a}} \right)^2 \Rightarrow \rho_{\text{dust}}^{(w)} = \frac{\rho_{0,d}}{a^3(1 + \sqrt{a_{\dagger}/a})^2}, \quad (64)$$

where  $a_{\dagger}$  is a constant ( $a_{\dagger} > a_*$ ).

### C. de Sitter expansion in the presence of matter

There is no better illustration of what the gauge freedom associated with scale invariance entails, than the following peculiar case. Let us investigate the possibility to have de Sitter expansion,  $H = H_0 \Rightarrow a = a_0 e^{H_0 t}$ , in the presence of standard matter. For definiteness we concentrate in the dust matter case. After the mentioned assumptions the Friedmann equation in (58) can be written in the following way:

$$H_0 + \frac{1}{2} \dot{\varphi} = h_0 e^{-(3H_0 t + \varphi/2)/2},$$

where  $h_0 \equiv \sqrt{\rho_{0,d}/3M_{\text{pl}}^2 a_0^3}$  and we have taken into account the Eq. (60). The above equation can be integrated and we obtain

$$e^{\varphi/2} = \left( \frac{h_0 a_0^{3/2}}{2H_0} \right)^2 \frac{(a-1)^2}{a^3}, \quad a(t) = a_0 e^{H_0 t}, \quad (65)$$

where we have appropriately chosen the integration constant  $C_0 = \ln(h_0/a_0)$ . Hence, given that the gauge scalar  $\varphi$  is related with the scale factor by Eq. (65), a matter source in the form of a dust fluid will lead to de Sitter expansion of the Universe. This means that in the present scale invariant setup we do not need the dark energy to produce late-time accelerated expansion. This possibility has no analogue in general relativity.

## VII. REDSHIFT

As a consequence of the point-dependent property of the energy  $E$  and masses of physical systems:

$$E(\varphi) \propto m(\varphi) = e^{\varphi/2} m_0,$$

the redshift of frequencies  $z \equiv \Delta\nu/\nu$  in the present setup may be explained, at least partly, as due precisely to varying mass/energy.<sup>13</sup> But in order to get the whole picture one have to invoke the null geodesics (44) which dictate the way the photon's frequency depends on the spacetime point.

For illustration, let us explore the cosmological redshift arising in a FRW spacetime, in which case the null-component of Eq. (44)

$$\frac{d\nu}{d\lambda} + H \frac{dx^0}{d\lambda} \nu = 0 \Rightarrow \frac{\dot{\nu}}{\nu} = -H,$$

<sup>13</sup> A similar discussion but within the so called “veiled general relativity” setup can be found in Ref. [32], while in [33] the point-dependent property of masses is associated with a so called “cosmon” field.

where we have taken into account that

$$k_\mu k^\mu = 0 \Rightarrow \mathbf{k}^2 = a^2 \delta_{ij} k^i k^j = \nu^2 = \frac{dx^0}{d\lambda} \nu,$$

can be readily integrated and we obtain:

$$\nu = \nu_0/a, \quad (66)$$

where  $\nu_0$  is a constant.

The point-dependent property of the masses of physical systems (like atoms, etc.), expressed through the equation (35), and the dependence of the photon's frequency upon the scale factor in (66) are enough to explain the redshift measured in cosmological observations of distant galaxies. Imagine two (identical) hydrogen atoms located in two separated galaxies:  $G$  – distant galaxy, and  $G_0$  – our galaxy. Suppose that due to a transition between the energy levels  $n_i$  and  $n_f$ , the hydrogen atom at the distant galaxy  $G$  emits a photon with frequency given by the Bohr's formula:<sup>14</sup>

$$\nu_G^E(\varphi) \equiv \nu_{n_i \rightarrow n_f}(\varphi) = \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \frac{m_e(\varphi) \alpha^2}{2h},$$

where  $h$  is the Planck's constant,  $\alpha$  is the fine structure constant, and  $m_e(\varphi) = e^{\varphi/2} m_e$  is the varying mass of the electron. It is apparent that the emitted frequency is a point-depnt quantity as well, however, once the photon leaves the atom, it follows a null geodesic given by Eq. (44), i. e., its dynamics is dictated by Eq. (66). Let us assume, for definiteness, that the Universe is filled with radiation, so that Eq. (62) takes place and

$$\nu_G^E = \frac{\mu_0}{2} \left( 1 + \frac{a_*}{a_E} \right), \quad \mu_0 \equiv \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \frac{m_e \alpha^2}{h}, \quad (67)$$

where for simplicity we assumed that the photon is emitted at a cosmic time  $t_E$ :  $a(t_E) = a_E = a_*$ , so that  $\nu_G^E = \mu_0$ . As long as the photon leaves the distant galaxy to reach to our galaxy  $G_0$  at present cosmic time  $t_0$ , its frequency obeys (see equation (66) with  $\nu_0 = \nu_G^E a_E$ )

$$\nu_{G \rightarrow G_0}^{\text{Obs}} = \nu_G^E \frac{a_E}{a_0} = \mu_0 \frac{a_E}{a_0}, \quad (68)$$

where  $a_0 = a(t_0)$  is the present value of the scale factor. Meanwhile, the frequency of the identical photon emitted by the hydrogen atom in our galaxy is given by

$$\nu_{G_0}^E = \left( 1 + \frac{a_E}{a_0} \right) \frac{\nu_G^E}{2}.$$

Hence, the resulting redshift in the photon's frequency is expressed through the following equation:

$$1 + z = \frac{\nu_{G_0}^E}{\nu_{G \rightarrow G_0}^{\text{Obs}}} = \frac{1}{2} \left( 1 + \frac{a_0}{a_E} \right). \quad (69)$$

The magnitude of the redshift depends on the background cosmic fluid. If instead of radiation consider dust, then due to Eq. (64) one would obtain

$$1 + z = \frac{1}{4} \left( 1 + \sqrt{\frac{a_0}{a_E}} \right)^2. \quad (70)$$

If the galaxy  $G$  is sufficiently far apart from ours  $G_0$ , i. e., if  $a_0 \gg a_E$ , since according to equations (62) and (64) at large  $a$ -s the gauge scalar  $\varphi = \varphi_0$  is a constant, then  $1 + z = a_0/\bar{a}_E$ , as in general relativity, where in the radiation dominated Universe we have conveniently rescaled  $a_E \rightarrow \bar{a}_E = 2a_E$  (in the dust case  $\bar{a}_E = 4a_E$ ).

### A. Redshift in a static Universe

A curious situation takes place if assume a constant scale factor in which case a static Universe is involved. In this case the ‘‘Friedmann equation’’ in (58) can be written as

$$\dot{\varphi}^2 = \frac{4\rho_{0\gamma}}{3M_{\text{pl}}^2} e^{-(3\gamma-2)\varphi/2},$$

where we have taken into account that the continuity equation for a barotropic fluid with equation of state  $P^{(w)} = (\gamma - 1)\rho^{(w)}$ , leads to  $\rho^{(w)} = \rho_{0\gamma} e^{-(3\gamma-2)\varphi/2}$ . Hence, according to the field equations the gauge scalar evolves in time:

$$\varphi(t) = \frac{4}{3\gamma - 2} \ln(\kappa_0 t), \quad \kappa_0 \equiv \frac{3\gamma - 2}{2} \sqrt{\frac{\rho_{0\gamma}}{3M_{\text{pl}}^2}}. \quad (71)$$

This means that, while the photon's frequency does not change during propagation [ $a(t) = \text{const.} \Rightarrow \nu = \nu_0$ ] the masses of elementary particles and composite systems do evolve in time  $m(t) = e^{\varphi(t)/2} m_0$ , which leads to a net redshift even if the Universe is not expanding at all. For illustration assume, as before, that an hydrogen atom at the distant galaxy  $G$  emits a photon of a given frequency

$$\nu_G^E = \frac{\mu_0}{2} e^{\varphi(t_E)/2},$$

at cosmic time  $t_E$ . Since once the photon is emitted its frequency does not change, the frequency emitted at  $G$  is the same observed in our galaxy  $G_0$  at latter cosmic time  $t_0$ :  $\nu_G^E = \nu_{G_0}^{\text{Obs}}$ . The observed frequency  $\nu_{G_0}^{\text{Obs}}$  is to be contrasted with the one emitted at cosmic time  $t_0$  by an identical hydrogen atom in our galaxy

$$\nu_{G_0}^E = \frac{\mu_0}{2} e^{\varphi(t_0)/2}.$$

<sup>14</sup> Recall that in this paper we use the units system where the speed of light  $c = 1$ .

Hence, even if the Universe is not expanding, a redshift arises

$$1 + z = e^{[\varphi(t_0) - \varphi(t_E)]/2} = \left(\frac{t_0}{t_E}\right)^{2/(3\gamma-2)}, \quad (72)$$

which depends on the cosmic fluid which fills the static spacetime. If the space is filled with radiation  $1 + z = t_0/t_E$ , while if the background cosmic fluid is dust then  $1 + z = (t_0/t_E)^2$ . In consequence the light coming from distant galaxies redshifts more during the present matter domination stage than in the past when radiation dominated the cosmic dynamics. This situation has no analogue within GR.

Although it seems weird to have redshift without expansion,<sup>15</sup> it should be noticed that in the present setup gravity is propagated both by the metric field and by the gauge scalar. Hence a static FRW spacetime does not mean absence of gravity but just that, in this particular case, gravity is entirely due to the gauge scalar, i. e., it is of pure affine origin.

### VIII. ACCELERATED EXPANSION WITHOUT EXPANSION

One of the current mysteries cosmologists face is related with the present (unexpected) speedup of the cosmic expansion which started in the recent past. In order to explain the accelerated expansion within the general relativity setup a certain exotic dark energy component is unavoidable [34]. Alternative explanations through modifications of Einstein's theory have been also considered [35].

This is not surprising that the observations on accelerated expansion can be explained even if the Universe is not expanding at all [33]. As shown in section VII A, within the present scale invariant setup there is room for redshift of light even in a static Universe. In this latter case the amount of redshift is intimately linked with the properties of the cosmic background. In particular, if the Universe is filled with radiation one have

$$1 + z = \frac{t_0}{t}, \quad (73)$$

where  $t_0$  is the present cosmic time and  $t$  is the time in the past at which light was emitted. Meanwhile, during a matter dominated period

$$1 + z = \left(\frac{t_0}{t}\right)^2. \quad (74)$$

In order to have accelerated expansion it is necessary that  $\ddot{a} > 0$ . Since in the present case we have no expansion at all, we are obliged to appeal to the relationship  $a(z) = 1/(1 + z)$ . Then

$$\ddot{a} > 0 \Rightarrow 2\dot{z}^2 > (1 + z)\ddot{z}. \quad (75)$$

If the FRW (static) WIG Universe is filled with radiation, taking time derivatives of (73) leads to

$$2\dot{z}^2 = (1 + z)\ddot{z},$$

so that the bound (75) is not satisfied. However, in a matter-dominated static WIG Universe, since

$$2\dot{z}^2 = \frac{8t_0^4}{t^6}, \quad (1 + z)\ddot{z} = \frac{6t_0^4}{t^6},$$

the bound (75) is indeed satisfied. In this latter case one observes a redshift pattern which is consistent with accelerated expansion even if the Universe is not expanding.

Although the static case studied above (see section VII A) may be considered just as a toy model which was useful to illustrate the possibilities of the present scale invariant theory, it has been shown in section VI C that, in general [ $a(t) \neq \text{const.}$ ] accelerated de Sitter expansion is possible in a FRW Universe filled with dust. Hence there is no need of any exotic dark energy component in this scale invariant setup in order to explain the late-time speedup of the cosmic expansion.

### IX. MASS HIERARCHY

Another nice result of the present study is that nothing besides the scale invariant theory (13) and inflation is required to explain the large hierarchy between the Higgs and Plack masses. Actually, suppose initial conditions are given in the neighborhood of the local maximum of the EW symmetry breaking potential  $V(|h|) = \lambda(|h|^2 - v_0^2(\varphi))^2/4$ , i. e., at  $h = 0$ . Let us also assume that  $\varphi(0) = 0$ , meaning that  $M_{\text{pl}}^2(0) = M_{\text{pl}}^2$ ,  $v_0^2(0) = v_0^2$ . The resulting theory is just EH de Sitter gravity:

$$S = \frac{M_{\text{pl}}^2}{2} \int d^4x \sqrt{|g|} \left( R - \frac{\lambda}{2} \frac{v_0^4}{M_{\text{pl}}^2} \right).$$

Since at  $h = 0$  the EW symmetry is unbroken, the particles of the SMP remain massless. Suppose, besides, that the initial value of the mass parameter is of the order of the Plack scale ( $v_0 \sim 10^{19}$  GeV), so that a large initial vacuum energy density  $\rho^{\text{vac}} = \lambda v_0^4/2 \sim M_{\text{pl}}^4$  (here we are considering the self-coupling  $\lambda \approx 1$ ) may fuel primordial inflation. After that the Universe starts inflating and the dynamics is described by the action (13).

To estimate the impact of inflationary expansion on the masses of SMP particles, let us assume that during the inflationary period the gravitational dynamics is approximately dictated by the WIG-Einstein equations

<sup>15</sup> See, however, Ref. [32].



$$G_{\mu\nu}^{(w)} = -\frac{\lambda}{4} \frac{v_0^4(\varphi)}{M_{\text{pl}}^2} e^{-\varphi} g_{\mu\nu} = -\frac{\lambda}{4} \frac{v_0^4}{M_{\text{pl}}^2} e^{\varphi} g_{\mu\nu}, \quad (76)$$

then, the relationship (57) takes place. As inflation proceeds, once the scale factor gets large enough  $a \gg a_0$ , i. e., once  $t \gg 1/H_0$  in (57), then  $e^{\varphi/2} \propto a^{-1}$ . The Higgs field settles down in the minimum of the symmetry breaking potential, and the SMP particles acquire point-dependent masses:

$$m_p(\varphi) \propto v_0(\varphi) = v_0 e^{\varphi/2}. \quad (77)$$

It immediately follows that inflation and point-dependence of mass are enough to explain the presently small mass of the SMP particles as compared with the Plack scale. Actually, suppose that during inflation the linear size of the Universe has expanded by a factor  $f = a_{\text{fin}}/a_{\text{ini}}$ . This means that during the inflationary stage the mass parameter would have decreased by the inverse factor:

$$\frac{v_0^{\text{fin}}}{v_0^{\text{ini}}} = \frac{a_{\text{ini}}}{a_{\text{fin}}} = f^{-1}.$$

Since the masses of the SMP particles are set by the mass parameter  $v_0^{\text{fin}}$ :

$$m_p \propto v_0^{\text{fin}} \propto f^{-1} v_0, \quad (78)$$

then, a modest inflationary factor  $f \sim 10^{16}$  is enough to explain the large hierarchy between the Higgs mass  $m_H \sim 1$  TeV and the Planck mass scale  $v_0 \sim M_{\text{pl}} \sim 10^{19}$  GeV. Since a factor of at least  $10^{27}$  is required to explain all of the puzzles inflation solves [36], this means that we do not need the entire inflationary epoch but just the final stages where the relationship  $a(\varphi) \propto e^{-\varphi/2}$  is approximately satisfied.

To complete the above explanation we recall that, as discussed in section V A, the actually measured value of the Cavendish gravitational constant is  $G = M_{\text{pl}}^{-2}/8\pi$  and not the very much larger (point-dependent)  $M_{\text{pl}}^{-2}(\varphi)/8\pi$ . Meanwhile, as shown in section V B, the masses of individual particles and of composite systems are usually measured through methods involving their interaction with electromagnetic fields in the presence of negligible background gravity as, for instance, in Thompson's experiment (also in mass spectrometers), so that it is the geodesic equation (50) (see also Eq. (52)) what matters, i. e., it is the varying mass  $m(\varphi) = e^{\varphi/2} m_0 \propto f^{-1} v_0$  and not the bare mass  $m_0 \propto v_0$ , the one which is measured in the laboratory.

By the same mechanism which generates the large hierarchy between the EW and the Plack energy scales, any existing dynamical vacuum energy – other than the one associated with the symmetry breaking potential – very quickly decays during inflation

$$\rho^{\text{vac}} = M_{\text{pl}}^4 e^{\varphi} \Rightarrow \rho_{\text{fin}}^{\text{vac}} \propto f^{-2} M_{\text{pl}}^4 \sim 10^{-32} M_{\text{pl}}^4. \quad (79)$$

This vacuum energy remaining after the end of inflation is larger than the presently accepted value by some 90 orders of magnitude. This means that, in order to explain the present stage of accelerated expansion of the Universe one have to renounce to an initially existing dynamical cosmological constant.

### A. Mass hierarchy and gauge freedom

We want to point out that the above explanation of the mass hierarchy is not gauge-independent. Recall that each different function,  $\varphi = \varphi(t)$ , singles out a specific gauge. If choose for instance the GR-gauge, where  $\varphi = 0$ , then the measured constants are the Newton's constant  $8\pi G = M_{\text{pl}}^{-2}$  ( $M_{\text{pl}} \propto 10^{19}$  GeV) and the bare rest mass  $m_0$  of about a TeV, so that the mass hierarchy issue can not be addressed. Another illustration is given by the vacuum solution with a cosmological constant term in section VI A. In that case the cosmic dynamics was specified by the choice  $a(t) \propto e^{H_0 t}$ , where, for convenience,  $H_0 = \sqrt{\Lambda/3}$ . The following relationship between the scale factor and the gauge scalar (see Eq. (57)):

$$e^{\varphi/2} = \frac{1}{1 + e^{H_0 t}},$$

was then singled out by the WIG-Friedmann equation (56). We can interpret the above choice another way around: one starts by choosing the latter functional dependence of  $\varphi$  vs  $t$  and the resulting de Sitter FRW metric  $g_{\mu\nu} = (-1, e^{2H_0 t} \delta_{ij})$  is obtained as a consequence of the cosmological field equations. In this gauge the cosmic expansion is inflationary while, for  $t \gg 1/H_0$ ,  $e^{\varphi/2} \sim e^{-H_0 t}$  falls off exponentially. This is, precisely, our explanation above for the small scale of the masses of the elementary particles which is generated after a period of cosmic inflation:

$$\frac{m_p^2}{M_{\text{pl}}^2} \propto e^{-2H_0 t_{\text{end}}},$$

where  $t_{\text{end}}$  is the approximate cosmic time marked by the end of inflation.

Let us now to choose a different gauge, say, the one where

$$e^{\varphi/2} \propto \alpha t \Rightarrow a(t) \propto \frac{e^{\sqrt{\Lambda/3} \alpha t^2/2}}{\alpha t},$$

where the same WIG-Friedmann equation (56) is satisfied. The above scale factor grows up exponentially faster than the de Sitter one [ $a(t) \propto e^{H_0 t}$ ] and, besides, it is never vanishing so that the resulting cosmic evolution is free of the big-bang singularity. Unfortunately in this case the exponential,  $e^{\varphi/2} \propto t$ , linearly increases with the

cosmic expansion so that the hierarchy of scales can not be generated through the mechanism explained above in this section. We can look for many other possible illustrations but we feel this is unnecessary.

Summarizing we can say that the mass hierarchy can be explained only in specific gauges by means of the approach followed in this section. There is a non-empty set of gauges in the class  $\mathcal{C}$  (32) where the mass hierarchy can not be explained. In the GR-gauge, in particular, this problem can not be even addressed, while in the gauge where  $\varphi \propto t$  the correct mass hierarchy is not generated.

## X. GAUGE-INDEPENDENT CURVATURE INVARIANTS AND THE SPACETIME SINGULARITIES

As discussed in section V, sometimes what one measures in experiments are physical quantities which are not gauge invariant. We have discussed this in the particular case of the gauge-dependent rest mass of the charged particles  $m(\varphi) = m_0 e^{\varphi/2}$  – see Eq. (35) – and also in the case of the Cavendish-like experiments, where it is the gauge covariant energy density  $\rho^{(w)}(\varphi) = e^{-\varphi} \rho$  – see Eq. (49) – the one which is measured by co-moving observers. Nonetheless, there are issues which require of gauge-independent geometric invariants to reach to gauge-independent conclusions. An outstanding example is the singularity issue. There has been a debate about the possibility that certain spacetime singularities in scalar-tensor theories – this includes the prototype Brans-Dicke theories [27] – can be avoided in their conformal formulations [4, 37, 38]. Unlike scalar-tensor theories which are not scale invariant, the present theory is gauge invariant so that the discussion of this issue is clearly different.

In order to discuss on the occurrence of spacetime singularities in our setup one is obliged to resort to the geometric invariants (45) which are not transformed by the scale transformations (11) and, hence, are the ones which carry gauge-independent physical meaning. As it was demonstrated in section III, general relativity is a particular gauge of the theory (13) when the Weyl scalar  $\varphi = \varphi_0$  is a constant where, without loss of generality, we set  $\varphi_0 = 0$ . Another way to see this is by realizing that under the Weyl rescaling (33), with  $\Omega^2 = e^\varphi$ :

$$\bar{g}_{\mu\nu} = e^\varphi g_{\mu\nu}, \quad \bar{\varphi} = 0,$$

the WIG-EH action (9)

$$S_{\text{EH}}^{(w)} = \frac{1}{2} \int d^4x \sqrt{|g|} M_{\text{pl}}^2 e^\varphi R^{(w)},$$

is mapped into the standard GR Einstein-Hilbert action

$$\bar{S}_{\text{EH}} = \frac{1}{2} \int d^4x \sqrt{|\bar{g}|} M_{\text{pl}}^2 \bar{R},$$

and vice versa. This entails that the following equalities involving the gauge-independent curvature invariants (45) are satisfied:

$$\begin{aligned} \bar{R} &= e^{-\varphi} R^{(w)}, \quad \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} = e^{-2\varphi} R_{\mu\nu}^{(w)} R^{\mu\nu}_{(w)}, \\ \bar{R}_{\mu\nu\kappa\lambda} \bar{R}^{\mu\nu\kappa\lambda} &= e^{-4\varphi} R_{\mu\nu\kappa\lambda}^{(w)} R^{\mu\nu\kappa\lambda}_{(w)}, \end{aligned} \quad (80)$$

where the quantities with an over-bar denote Riemannian objects defined in terms of the Christoffel symbols of the metric  $\bar{g}_{\mu\nu}$ . Notice that equalities like the ones in (80) arise only in gauge invariant theories where the gauge-independent curvature invariants (45) make sense. These are not necessarily satisfied when dealing with standard scalar-tensor theories like the BD-theory [27].

In what follows for simplicity of writing we shall identify the following curvature invariants:

$$I_1 \equiv R, \quad I_2 \equiv R_{\mu\nu} R^{\mu\nu}, \quad I_4 \equiv R_{\mu\nu\kappa\lambda} R^{\mu\nu\kappa\lambda}. \quad (81)$$

Hence, for instance,  $\bar{I}_1 = \bar{R}$ ,  $I_2^{(w)} = R_{\mu\nu}^{(w)} R^{\mu\nu}_{(w)}$ , etc. The equations (80) can then be written in a more compact form:

$$\bar{I}_1 = e^{-\varphi} I_1^{(w)}, \quad \bar{I}_2 = e^{-2\varphi} I_2^{(w)}, \quad \bar{I}_4 = e^{-4\varphi} I_4^{(w)}, \quad (82)$$

correspondingly. Let us assume the following hypothetical situation: the GR spacetime  $(\mathcal{M}, \bar{g}_{\mu\nu})$  has a curvature singularity at some point  $x_P$  in the manifold, such that the GR invariant  $\bar{I}_4 = \alpha(x)$ , blows up at  $x_P$ , i. e.,

$$\lim_{x \rightarrow x_P} \alpha(x) \rightarrow \infty. \quad (83)$$

Let us further assume that

$$\lim_{x \rightarrow x_P} \varphi(x) \rightarrow -\infty \Rightarrow \lim_{x \rightarrow x_P} e^{4\varphi(x)} \rightarrow 0,$$

where the latter limit is approached quicker than the one in Eq. (83), such that the WIG-curvature invariant  $I_4^{(w)}$  is finite at  $x_P$ . This means that, given that the conditions of the hypothetical situation described above are fulfilled, the GR curvature singularity is not felt by an observer living in the conformal WIG-world. This does not mean that the singularity has been erased by the Weyl rescaling (33). As a matter of fact the physically meaningful gauge-independent curvature invariant  $e^{-4\varphi} I_4^{(w)}$  blows up at  $x_P$ , meaning that the singularity is still there. To understand what actually happens when the singularity is approached one have to recall that in WIG spacetimes the units of measure are point-dependent. As the singularity is approached and the given gauge-independent curvature invariants grow up without bound, the corresponding units of measure increase in a similar fashion so that the increase in the magnitude of the invariants is conveniently balanced.

For a quantitative analysis it is convenient to study a concrete example. Here we shall consider as an specific

example the GR cosmological singularity usually associated with the big-bang. As before, the quantities with an over-bar are referred to Riemannian GR quantities. We have

$$d\bar{s}^2 = -d\bar{t} + \bar{a}^2(\bar{t})d\mathbf{x}^2, \quad ds^2 = -dt + a^2(t)d\mathbf{x}^2,$$

where the left-hand line-element refers to the (conformal) GR-FRW spacetime, while the right-hand one refers to WIG-FRW spacetime. The following relationships arise:

$$dt = e^{-\varphi/2}d\bar{t}, \quad a(\bar{t}) = e^{-\varphi/2}\bar{a}(\bar{t}), \quad \rho^{(w)} = e^\varphi\bar{\rho}, \quad (84)$$

where  $\rho^{(w)}$  is the energy density of matter measured by a co-moving observer in the WIG-world, and we assumed the conformal factor  $\Omega^2 = e^\varphi$ . Consider a singular GR solution  $\bar{a} \propto \bar{t}^n$  ( $n$  is an arbitrary positive constant), so that according to the GR-Friedmann constraint [ $3\bar{H}^2 = \bar{\rho}/M_{\text{pl}}^2$ ], we have  $\bar{\rho} \propto \bar{t}^{-2}$ . The initial cosmological singularity is at  $\bar{t} = 0$  where the matter energy density  $\bar{\rho}$  blows up. It is a simple exercise to show that invariants  $\bar{I}_1$ ,  $\bar{I}_2$  and  $\bar{I}_4$  go to infinity at the singularity. In particular  $\bar{I}_1 \propto \bar{H}^2 \propto \bar{t}^{-2}$  grows up without bound as  $\bar{t} \rightarrow 0$ . Among the infinity of possibilities let us choose

$$\varphi(\bar{t}) = \ln \left( \frac{\cosh^2 \bar{t} - 1}{\cosh^2 \bar{t}} \right). \quad (85)$$

Recall that due to the gauge freedom we are free to choose any  $\varphi$  we want (do not forget certain trivial mathematical requirements). After the above convenient choice one gets that the WIG curvature invariant

$$I_1^{(w)} = e^\varphi \bar{I}_1 \propto \frac{\cosh^2 \bar{t} - 1}{\bar{t}^2 \cosh^2 \bar{t}},$$

is always bounded ( $0 \leq \bar{t} < \infty$ ). The same is true for the energy density measured by a co-moving WIG-observer  $\rho^{(w)} = e^\varphi \bar{\rho} \propto I_1^{(w)}$ . Hence a co-moving observer in the equivalent WIG picture does not find singular behavior at all. As already stated, the explanation is simple: although the singularity is still there, as the co-moving observer approaches to it, any unbounded increment in any of the gauge-independent curvature invariants is balanced by a proportional increment in the corresponding units of measure in the WIG-FRW spacetime.

Regarding the amount of cosmic time separating a given co-moving observer from the singularity we have to say that, while the interval of cosmic time from, say, the present moment of the cosmic history  $\bar{t}_0$  to the singularity in the past at  $\bar{t} = 0$ , is finite, in terms of the cosmic time measured by an observer in the WIG-world we have that

$$\Delta t = \int_\epsilon^{\bar{t}_0} \frac{d\bar{t} \cosh \bar{t}}{\sqrt{\cosh^2 \bar{t} - 1}} = \ln \left( \frac{\sinh \bar{t}_0}{\sinh \epsilon} \right),$$

where  $\epsilon$  is a small number such that, as one approaches to the initial singularity,  $\epsilon \rightarrow 0$ . Hence, as  $\epsilon \rightarrow 0$ ,  $\Delta t \rightarrow \infty$ .

This means that to an observer in the WIG-world the initial (big-bang) singularity is an infinite amount of cosmic time into the past. In consequence, for all practical purposes the corresponding WIG geodesics are complete into the past and the initial singularity – although not erased – is effectively avoided. The conclusion is that the singularity issue, just like the mass-hierarchy issue, is gauge-dependent.

## XI. GENERIC WEYL-INVARIANT THEORIES OF THE SMP AND GRAVITY

In this section, motivated by the fact that a certain degree of confusion may arise, we shall discuss on the profound differences between the scale invariant theory (13) which is grounded in WIG spacetimes, and the generic Weyl-invariant standard model which was investigated in Ref. [18] (see also [20–22]). It is depicted by the action (8). In the mentioned theory, in order to allow for geodesic completeness it is required that not only the dilaton field  $\phi$ , but also the doublet Higgs field, be a set of conformally coupled scalars consistent with  $SU(2) \times U(1)$ , namely using the conformally invariant unit

$$\frac{1}{6} |H|^2 R + |D^* H|^2,$$

where  $D^*$  is the gauge covariant derivative defined in (7), in conjunction with

$$\frac{1}{12} \phi^2 R + \frac{1}{2} (\partial\phi)^2.$$

The problem with the approach of [18, 20–22] is that nothing is said about the affine geometrical structure of the underlying manifold or about whether the Weyl symmetry of the action is shared by the geometrical laws. That this poses an actual problem for theories claimed to be Weyl-invariant will be evident from the following discussion. Let us consider a Weyl-invariant extension of theories of gravity with two scalar fields coupled to the curvature which is given by the action [22]:

$$S = \int d^4x \sqrt{|g|} \left\{ \frac{1}{12} (\phi^2 - \sigma^2) R + \frac{1}{2} [(\partial\phi)^2 - (\partial\sigma)^2] \right\}, \quad (86)$$

where the gravitational coupling  $8\pi G$  is replaced by  $6/(\phi^2 - \sigma^2)$ . The above action is invariant under the Weyl gauge transformations

$$g_{\mu\nu} \rightarrow \Omega^{-2} g_{\mu\nu}, \quad \phi \rightarrow \Omega\phi, \quad \sigma \rightarrow \Omega\sigma. \quad (87)$$

For the coupling  $\propto (\phi^2 - \sigma^2)^{-1}$  to be positive and the theory Weyl-invariant, the scalar  $\varphi$  must have a wrong

sign kinetic energy, potentially making it a ghost. However, the local Weyl gauge symmetry compensates, thus ensuring the theory is unitary [22].

Now we shall show that, unless the affine geometrical structure of the conformal spacetime manifold is specified and the impact of the Weyl gauge transformations (87) on the underlying geometrical laws is appropriately discussed, there is an ambiguity in the understanding of the scale invariance associated with the invariance of the action (86) under (87).

### A. Riemannian background spacetimes

Let us start by adopting that the affine properties of the spacetimes which solve the field equations derived from (86) are governed by the Riemann geometric theory. This means, in particular, that the affine connection coincides with the Christoffel symbols of the metric and that the Riemann metricity condition is satisfied:  $\nabla_\mu g_{\alpha\beta} = 0$ . Under the conformal transformation of the metric in (87) the Christoffel symbols transform like

$$\{\alpha_{\beta\sigma}\} \rightarrow \{\alpha_{\beta\sigma}\} - \frac{1}{\Omega} (\delta_\beta^\alpha \partial_\sigma \Omega + \delta_\sigma^\alpha \partial_\beta \Omega - g_{\beta\sigma} \partial^\alpha \Omega). \quad (88)$$

In this case it is understood that, given that Riemann geometry governs the affine properties of the original spacetime, the affine structure of the conformal space is also Riemannian, i. e., the Riemannian metricity condition

$$\nabla_\mu g_{\alpha\beta} = 0 \rightarrow \nabla_\mu g_{\alpha\beta} = 0,$$

is preserved by the conformal transformation of the metric, so that  $(\mathcal{M}, g_{\mu\nu}) \rightarrow (\mathcal{M}, g_{\mu\nu})$ . Notice that while the laws of gravity represented by (86) are unchanged by (87), the Riemannian geodesics

$$\frac{d^2 x^\alpha}{ds^2} + \{\alpha_{\mu\nu}\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0, \quad (89)$$

are mapped into non-geodesics of the conformal space [4]

$$\frac{d^2 x^\alpha}{ds^2} + \{\alpha_{\mu\nu}\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = \frac{\partial_\mu \Omega}{\Omega} \frac{dx^\mu}{ds} \frac{dx^\alpha}{ds} - \frac{\partial^\alpha \Omega}{\Omega}, \quad (90)$$

where under a convenient re-parametrization,  $ds \rightarrow d\sigma = \Omega ds$ , the first term in the RHS can be eliminated, but the second term  $\propto -\Omega^{-1} \partial^\alpha \Omega$  can not be eliminated at all. The 2nd term in the RHS of (90) can be understood as an additional “5-force” of non-gravitational origin [4]. This additional non-gravitational force is the responsible for the avoidance of certain spacetime singularities which are present in the original spacetime but which are not met in the conformal one. In other words, incomplete geodesics in the original spacetime can be mapped into complete (non)geodesics in the conformal space thanks to the 5-force  $\propto -\Omega^{-1} \partial^\alpha \Omega$ , which deviates the motion of a point-particle from being geodesic. The role of the conformal

transformations in this case is to send the singularity to one of the ends of the (complete) non-geodesic curve at infinity. I. e., the singularity is still there but it takes an infinite proper (conformal) time along one such non-geodesic curve to reach to it. Notice, however, that the geodesic equations for massless particles are not affected by the conformal transformation of the metric, so that photons always see the singularity (see [38] for a related discussion).

A similar situation can be associated with the conservation of stress-energy. Suppose in the original field variables:

$$\nabla^\mu T_{\mu\nu}^{(m)} = 0, \quad (91)$$

where  $T_{\mu\nu}^{(m)}$  is the stress-energy tensor of matter. Under the conformal transformation in (87) the above conservation equation is mapped into

$$\nabla^\mu T_{\mu\nu}^{(m)} = -\frac{\partial_\nu \Omega}{\Omega} T^{(m)}, \quad (92)$$

where  $T^{(m)} = g^{\mu\nu} T_{\mu\nu}^{(m)}$  is the trace of the SET of matter. The source term in the RHS of (92) is to be associated with the mentioned 5-force. It is due to the well-known fact that under a conformal transformation of the metric the original matter Lagrangian acquires a non-minimal coupling to the conformal factor [1]:  $\mathcal{L}_{\text{mat}} \rightarrow \Omega^{-4} \mathcal{L}_{\text{mat}}$ . Physically Eq. (92) is interpreted in the following way. In the conformal variables the balance of stresses and energy requires of a certain matter flux to compensate the effect of the 5-force. Only for traceless matter the standard conservation equation is satisfied also in the conformal field variables. It is usually said that a non-vanishing trace spoils any existing Weyl invariance when matter is considered. This is a very well-known fact which is missing in the discussion of [18, 20–22].

We are faced with the following perturbing paradox: while the gravitational Einstein’s equations derived from (86) are not modified by (87), the laws of motion of the non-gravitational fields which are not conformally coupled to gravity, e. g., dust, which are expressed through the geodesics and/or through the continuity equations for the corresponding field, are in general transformed by the Weyl gauge transformations. Recall that the geodesic curves are indeed modified by (87) – geodesics in the original frame (89) are transformed into non-geodesics (90) in the conformal frame – as well as the conservation of energy and stresses (91) which transforms into Eq. (92), where it is evident that the energy-stresses are not conserved in the conformal frame due to the dissipation associated with the required compensation of the effects of the 5-force. The bad thing here is that there are very stringent constraints on the 5-force [28] which have to be satisfied by (86) in any of its conformal formulations provided that in the original formulation it is not present.

Another important feature of the theory (86) when associated with Riemannian manifolds resides in that the

gravitational field equations derived from (86): the Einstein's equations plus the Klein-Gordon equations for the scalar fields  $\phi$  and  $\sigma$ , are enough to pick specific solutions,  $g_{\mu\nu}^{\text{sol}}(x)$ ,  $\phi_{\text{sol}}(x)$ ,  $\sigma_{\text{sol}}(x)$ , which amounts to pick specific spacetimes  $(\mathcal{M}, g_{\mu\nu}^{\text{sol}})$  together with their matter content  $(\phi_{\text{sol}}(x), \sigma_{\text{sol}}(x))$ . Hence, there is not any conformal degree of freedom in the choice of the spacetimes which solve the field equations, a feature which one should expect to arise in truly conformal-invariant theories.

To put the above feature of the theory of [20–22] in perspective, suppose we start with the action (86) where it is assumed that point particles follow geodesics of the Riemann geometry (89) and that the conservation of the energy and stresses (91) is satisfied. Assume also that  $[g_{\mu\nu}^{\text{sol}}(x), \phi_{\text{sol}}(x), \sigma_{\text{sol}}(x)]$  is a solution of the field equations which are derived from (86). The Weyl gauge transformations (87) send the original theory into a theory where the gravitational laws are the same (the same action), but where the point-particles do not move in Riemannian geodesics but in paths which are acted on by an additional non-gravitational force (90), a fact which is also reflected in the continuity equation for the matter stress-energy tensor (92). Simultaneously, the Weyl gauge transformations send the assumed solution into a corresponding solution of the conformal theory:  $[\Omega^2 g_{\mu\nu}^{\text{sol}}(x), \Omega^{-1} \phi_{\text{sol}}(x), \Omega^{-1} \sigma_{\text{sol}}(x)]$ . The lack of gauge freedom is distinctive of any supposedly Weyl-invariant theory where the geometrical structure of the spacetime does not share the Weyl invariance of the action. This is a delicate issue which is most times ignored.

We can see that the theory (86) comprises, as a matter of fact, a whole class of different, actually non-equivalent conformal theories: Assuming, for instance, that the original formulation is free of the 5-force, this unwanted ingredient will be present in any of the conformal formulations of the theory with a different strength. Hence, in principle, the experiment may differentiate among the different conformal theories depicted by (86). In this case, given that under (87) the original theory is mapped into a different conformal theory, it is not clear what to understand by Weyl gauge invariance, besides plain gauge invariance of the action.

## B. WIG background spacetimes

In the former sections we have shown that in an actually Weyl-invariant theory, as the one given by (13) with the specification of the WIG structure of the space, the field equations alone are not enough to uncover the dynamics of the spacetime, i. e., in addition to the usual degrees of freedom to make spacetime diffeomorphisms a gauge degree of freedom to make conformal transformations arises. In spite of the redundancy, in what follows we shall explore yet another example of such an actual Weyl invariant theory which will make evident the differences between our setup and the generic theories of the kind (86) [18, 20–22].

We shall start precisely with the action (86) but, additionally, we will postulate that the spacetimes which solve the derived field equations have WIG affine structure in place of the Riemannian structure assumed in the former subsection. In order to make the theory (86) compatible with the former postulate we shall assume that the field  $\phi$  is the Weyl gauge scalar, so that it is lifted to the category of a geometric field in addition to the metric field itself  $g_{\mu\nu}$ . The corresponding WIG affine connection of the manifold is defined as

$$\Gamma_{\beta\mu}^\alpha = \{\}_{\beta\mu}^\alpha + \frac{1}{\phi} (\delta_\beta^\alpha \partial_\mu \phi + \delta_\mu^\alpha \partial_\beta \phi - g_{\beta\mu} \partial^\alpha \phi). \quad (93)$$

Besides, the standard derivatives of non-geometric fields like  $\sigma$ , should be replaced by

$$\partial_\mu \sigma \rightarrow D_\mu \sigma \equiv \left( \partial_\mu - \frac{\partial_\mu \phi}{\phi} \right) \sigma.$$

After the above specifications, the resulting Weyl invariant action can be written as follows [compare with (86) to see the subtle differences]:

$$S^{(w)} = \int d^4x \sqrt{|g|} \left\{ \frac{1}{12} (\phi^2 - \sigma^2) R^{(w)} - \frac{1}{2} (D\sigma)^2 \right\} \quad (94)$$

where the different quantities and operators are defined in terms of the WIG affine connection (93). This apparently slight modification of (86) designed to make that theory compatible with WIG backgrounds, results in that the Klein-Gordon equation for  $\phi$  is not an independent equation anymore but it coincides with the trace of the Einstein equations. In consequence a gauge degree of freedom arises due to Weyl symmetry and is reflected in that the field equations are not enough to explicitly determine the metric and the Weyl gauge scalar at once. These equations amount just to a given relationship among the geometric fields  $g_{\mu\nu}$  and  $\phi$ .

In this Weyl invariant modification of (86) which is grounded in WIG backgrounds, not only the field equation derived from (94), but also the WIG-geodesics

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma_{\mu\nu}^\alpha \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = \frac{\partial_\mu \phi}{\phi} \frac{dx^\mu}{ds} \frac{dx^\alpha}{ds},$$

or after a convenient re-parametrization  $d\sigma = \phi ds$ ,

$$\frac{d^2 x^\alpha}{d\sigma^2} + \Gamma_{\mu\nu}^\alpha \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} = 0, \quad (95)$$

and the WIG continuity equation

$$\nabla_{(w)}^\mu T_{\mu\nu}^{(m)} = 2 \frac{\partial^\mu \phi}{\phi} T_{\mu\nu}^{(m)}, \quad (96)$$

where the covariant derivative operator  $\nabla_{(w)}^\mu$  is defined in terms of the WIG affine connection  $\Gamma_{\beta\mu}^\alpha$ , all are invariant

under the Weyl gauge transformations (87). In equation (96) the term in the RHS is not actually a source term, instead it expresses the fact that in WIG spaces the units of measure of energy and stresses are point-dependent quantities, as well as the length of any other vector. The resulting picture is similar to the one arising in our setup: there is an infinite set of WIG spacetimes  $(\mathcal{M}, g_{\mu\nu}, \phi)$  which are related by Weyl gauge transformations (87), which satisfy the same set of motion equations: the field equations derived from (94), the geodesic equations (95), the continuity equation (96), etc. The different gauges in the above set amount to equivalent geometrical descriptions of the given laws of gravity.

As we have seen, in addition to postulating the laws of gravity by means of a given choice of the action, say (86), a separate postulate on the geometric properties of the underlying spacetime manifold is mandatory. Besides it is convenient to discuss how the Weyl gauge transformations impact the geometric properties of the underlying manifold. These include the geodesics and the continuity equation which govern the dynamics of the interaction of matter with the geometry. Otherwise the ambiguity in the geometrical interpretation of the given laws of gravity raises any kind of miss-interpretations, paradoxes and unnecessary confusion.

Before closing this section we want to point out another crucial difference of our setup from the theory (86) of [20–22]. In addition to the obvious (geometrical) differences between (86) and (94) which have been discussed above, our theory could be recovered from the latter action – which is not the same as in (86) thanks to the different underlying affine geometrical structure – if further remove the non-minimal coupling between the non-geometric field  $\sigma$  (in our setup it is the Higgs field) and the WIG curvature scalar in (94), i. e., if remove the term  $\propto \sigma^2 R^{(w)}$ . That this is a clear difference of the setup of [18, 20–22] from ours can be seen if realize that in the theory (86) the replacement of the (inverse) gravitational coupling by the combination of two (or more) scalar fields in the form  $\propto (\phi^2 - \sigma^2)$  is cornerstone to discuss on geodesic completeness.

## XII. DISCUSSION AND CONCLUSION

Although most theories of the fundamental interactions (including general relativity and string theory) assume that the geometric structure of spacetime is pseudo-Riemann, there are indications that a generalization of Riemann geometry – assumed here to be Weyl integrable geometry – might represent a better suited arena where to formulate the laws of gravity [1, 4, 5, 9, 11, 12].

The setup we have explored here combines gravity and the SMP into a scale invariant scheme sustained by spacetimes whose affine properties are governed by Weyl integrable geometry, which is capable of explaining the large hierarchy between the Planck and Higgs scales as a by-product of cosmological inflation. The additional degree

of freedom introduced by scale invariance provides also an alternative explanation of the observational evidence on late-time accelerated expansion of the Universe. Due to the fact that in the present theory both the metric field  $g_{\mu\nu}$  and the gauge scalar  $\varphi$  propagate gravity, then even in a static cosmological background a certain redshift arises which is associated with the point-dependent property of particles masses  $m(\varphi) \propto e^{\varphi/2}$ . The resulting redshift depends on the properties of the cosmological background in such a way that the amount of redshift is larger in a matter-dominated Universe than in the radiation domination stage. In particular, as shown in section VIII, a redshift consistent with evidence on accelerated expansion may be obtained if the static Universe is filled with dust.

A distinctive feature of our setup is that not only the action of the theory (13) – and the corresponding field equations – but also the WIG non-metricity condition  $\nabla_{\mu}^{(w)} g_{\nu\lambda} = -\partial_{\mu}\varphi g_{\nu\lambda}$ , the WIG geodesic equations

$$\frac{d^2 x^{\alpha}}{ds^2} + \Gamma_{\mu\lambda}^{\alpha} \frac{dx^{\mu}}{ds} \frac{dx^{\lambda}}{ds} = \frac{1}{2} \partial_{\lambda}\varphi \frac{dx^{\lambda}}{ds} \frac{dx^{\alpha}}{ds},$$

and, outstandingly, the conservation equation

$$\nabla_{(w)}^{\lambda} T_{\lambda\mu}^{(w,m)} = 0,$$

where  $T_{\mu\nu}^{(w,m)}$  is the gauge invariant stress-energy tensor of matter, all are invariant under the Weyl gauge transformations (11). This means that not only the action of the theory but also the geometrical structure under which it rests, are Weyl gauge invariant. Accordingly we can properly say that the resulting theory (= action + geometrical structure) is Weyl gauge invariant or, simply, scale invariant. Given that thanks to the Weyl gauge invariance the field equations are not enough to solve for the spacetime dynamics of the geometric fields  $g_{\mu\nu}$  and  $\varphi$ , one is free to choose either  $\varphi = \varphi(x)$  or one of the ten components of the metric, say,  $g_{nm} = g_{nm}(x)$ . Hence, there is not one single solution/gauge  $(g_{\mu\nu}^{\text{sol}}(x), \varphi_{\text{sol}}(x))$  which satisfies the WIG-Einstein equation

$$G_{\mu\nu}^{(w)} = 8\pi G T_{\mu\nu}^{(w,m)},$$

plus the motion equations of the remaining matter degrees of freedom (Klein-Gordon equations, etc.), but a whole class of them

$$\mathcal{C} = \{(\Omega_k^2 g_{\mu\nu}^{\text{sol}}(x), \varphi_{\text{sol}}(x) - \ln \Omega_k^2) : \Omega_k^2 = \Omega_k^2(x), k = 1, 2, \dots, \infty\},$$

where the  $\Omega_k^2(x)$  are any positive, non-vanishing smooth real-valued functions. What the gauge freedom means is that the field equations are not enough to pick one specific gauge in  $\mathcal{C}$ . In section IX A we have shown, in a cosmological setting, that there might be clues that allow one to rule out a large number of gauges in the

above equivalence class. It was demonstrated, in particular, that if the mass hierarchy problem is to be solved as a byproduct of inflation in our setup, there are several gauges which are ruled out (the GR-gauge is one of them). What this means is that the resolution of the mass hierarchy problem – as well as, perhaps, many other fundamental problems of physics – is not gauge-independent. This led us to conjecture in the most speculative part of section III, that there exist infinitely many different gauges/spacetimes comprised in our scale invariant setup but our own existence, which means that we can perform experiments and do observations, picks one specific gauge: the one which allows a correct description of the existing amount of observational/experimental evidence. This picture shares certain resemblance with the multiverse scenario [23].

Another encouraging feature of the present scale invariant theory of gravity and SMP – not explored in this paper – is related with the possibility of quantization. Actually, thanks to the additional degree of freedom to make scale transformations (11), one can have gravitation even in Minkowski spacetime  $g_{\mu\nu} = \eta_{\mu\nu}$ . In this particular gauge the gravitational interactions are of affine origin exclusively and are propagated by the gauge scalar  $\varphi$  alone. The point is that we know how to quantize scalar fields in Minkowski background so that, by exploring this particular gauge [ $g_{\mu\nu} = \eta_{\mu\nu}$ ,  $\varphi = \varphi(\mathbf{x})$ ], we could obtain important insights into the quantum properties of gravity. Exploration of this possibility will be the subject of forthcoming work.

Although scale invariance of the fundamental laws of physics has been the subject of intensive and longstanding debate [14–18, 20–22], the general consensus is that this invariance is spoilt by the Higgs mechanism for generating masses. As stated for instance in [17] (see also [18, 21]), it is usually understood that in a scale invariant theory there are no mass scales and the hope is that when the symmetry is broken the large hierarchies between the mass scales emerge naturally. In the present paper we have demonstrated through a concrete example that scale invariance and the Higgs mechanism are perfectly compatible and that the hierarchy issue can be solved without renouncing to scale invariance.

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### XIII. APPENDIX

#### A. Weyl-integrable geometry

It is known since long ago that a drawback of Weyl's geometric theory is associated with non-integrability of length in this theory. Actually, under parallel transport of a vector  $\mathbf{l}$  along a closed path in spacetime, its length  $l = |\mathbf{l}|$  is changed according to:  $l = l_0 \exp \oint dx^\mu w_\mu / 2$ . This might be associated with an unobserved broaden-

ing of the atomic spectral lines, also known as the “second clock effect” [5]. There is a simpler variant of Weyl geometry called as “Weyl integrable geometry” (WIG), which is free of the mentioned problem. WIG is obtained from Weyl theory if make the replacement  $w_\mu \rightarrow \partial_\mu \varphi$ , where  $\varphi$  is known as the Weyl gauge scalar. In this case, since  $\oint dx^\mu \partial_\mu \varphi / 2 = 0$ , then the length of a vector is integrable. Although several authors consider the above replacement as a trivial gauge and identify the resulting geometry with standard Riemann space, this is wrong. In fact, in the obtained affine structure, the (integrable) lengths of vectors are actually point-dependent. As a result, the affine connection (10):

$$\Gamma_{\beta\gamma}^\alpha = \{\}_{\beta\gamma}^\alpha + \frac{1}{2} (\delta_\beta^\alpha \partial_\gamma \varphi + \delta_\gamma^\alpha \partial_\beta \varphi - g_{\beta\gamma} \partial^\alpha \varphi),$$

the non-metricity condition of WIG

$$\nabla_\mu^{(w)} g_{\alpha\beta} = -\partial_\mu \varphi g_{\alpha\beta}, \quad (97)$$

the corresponding WIG Riemann-Christoffel and Ricci tensors, and the covariant derivative operator

$$R_{\alpha\beta\mu\nu}^{(w)}, R_{\mu\nu}^{(w)}, \nabla_\mu^{(w)},$$

among other quantities, are all invariant under the following local scale transformations/Weyl rescalings (11):

$$g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}, \quad \varphi \rightarrow \varphi - 2 \ln \Omega.$$

This means that there is not a single Weyl integrable space  $(\mathcal{M}, g_{\mu\nu}, \varphi)$ , but a whole equivalence class of them (32):

$$\mathcal{C} = \{(\mathcal{M}, g_{\mu\nu}, \varphi) : \nabla_\mu^{(w)} g_{\alpha\beta} = -\partial_\mu \varphi g_{\alpha\beta}\},$$

such that any other pair  $(\bar{g}_{\mu\nu}, \bar{\varphi})$  related with  $(g_{\mu\nu}, \varphi)$  by a scale transformation (11), also belongs in the conformal equivalence class  $\mathcal{C}$ . This property is not shared by (pseudo)Riemann geometry which corresponds to the particular GR gauge:  $\varphi = \varphi_0 = \text{const.}$

#### B. Scale invariance of the EW Lagrangian

Here we present the demonstration given in [12] that the EW Lagrangian terms not included in (13) do not spoil the local scale invariance of this theory. Let us start with the action piece corresponding to the gauge fields

$$S_{\text{gauge}} = -\frac{1}{4} \int d^4x \sqrt{|g|} (W_{\mu\nu}^k W_k^{\mu\nu} + B_{\mu\nu} B^{\mu\nu}),$$

where  $W_{\mu\nu}^k \equiv \partial_\mu W_\nu^k - \partial_\nu W_\mu^k$  and  $B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu$  are usual field strengths for the SU(2) and U(1) gauge bosons. As long as the gauge vectors  $W_\mu^k$  and  $B_\mu$  are

unchanged by the Weyl rescalings (11), the above Lagrangian density is also scale invariant. The typical action for a fermion field  $\psi$  in EW theory in a curved (pseudo)Riemann background is given by

$$S_\psi = i \int d^4x \sqrt{|g|} \bar{\psi} \gamma^c e_c^\mu \left[ \hat{D}_\mu - \frac{1}{2} \sigma_{ab} e^{b\nu} (\partial_\mu e_\nu^a - \{\lambda_{\mu\nu}^\lambda\} e_\lambda^a) \right] \psi,$$

where  $e_\mu^a$  is the tetrad and

$$\hat{D}_\mu^* \psi = \left( \partial_\mu + ig W_\mu^k T^k - \frac{i}{2} g' Y B_\mu \right) \psi,$$

is the covariant gauge derivative, with  $T^k$  the isospin matrices and  $Y$  the fermion's hypercharge. To make the above action  $S_\psi$  not only  $SU(2) \otimes U(1)$  gauge invariant but, also, invariant under the Weyl rescalings (11), the following replacements are to be made:

$$\begin{aligned} \hat{D}_\mu^* &\rightarrow \hat{D}_\mu = \hat{D}_\mu^* - \frac{3}{4} \partial_\mu \varphi, \\ \partial_\mu e_\nu^a &\rightarrow \left( \partial_\mu + \frac{1}{2} \partial_\mu \varphi \right) e_\nu^a, \quad \{\lambda_{\beta\gamma}^\alpha\} \rightarrow \Gamma_{\beta\gamma}^\alpha, \end{aligned}$$

where the WIG affine connection  $\Gamma_{\beta\gamma}^\alpha$  is given in Eq. (10). Besides, under (11) the fermion and tetrad fields transform like

$$\psi \rightarrow \Omega^{3/2} \psi, \quad e_\mu^a \rightarrow \Omega^{-1} e_\mu^a, \quad (98)$$

respectively. It is a matter of straightforward algebra to show that all terms  $\propto \partial_\mu \varphi$  cancel out. This means that fermions do not couple to the (gradient of the) gauge scalar field  $\varphi$ . Hence, the action

$$S_{\text{tot}} = S^{(w)} + S_{\text{gauge}} + \sum S_\psi,$$

is not only scale invariant but, also, it perfectly accommodates the SMP.

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